

Search and Rest Unemployment*

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Abstract

This paper develops a tractable version of the [Lucas and Prescott \(1974\)](#) search model. Each of a continuum of industries produces a heterogeneous good using a production technology that is continually hit by idiosyncratic shocks. In response to adverse shocks, some workers search for new industries while others are rest unemployed, waiting for their industry's condition to improve. We obtain closed-form expressions for key aggregate variables and use them to evaluate the model's quantitative predictions for unemployment and wages. Both search and rest unemployment are important for understanding the behavior of wages at the industry level.

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1 Introduction

This paper develops a tractable version of [Lucas and Prescott's \(1974\)](#) theory of unemployment. We distinguish between “search” and “rest” unemployment. Search unemployment is a costly reallocation activity whereby a worker attempts to move to a better industry. Rest unemployment is a less costly activity whereby a worker waits for her current industry’s condition to improve. We obtain closed-form solutions for the search and rest unemployment rates in terms of reduced-form parameters, which are in turn known functions of preferences and technology. The dynamics of industry wages are determined by the same reduced-form parameters, giving us a tight link between aggregate unemployment and the stochastic process for industry-level wages. We exploit that link to evaluate quantitatively the role played by search and rest unemployment in our model.

Our model economy consists of a continuum of industries, each with many workers and firms. Firms in each industry produce an intermediate good with a constant returns to scale technology using labor only. The growth rate of output per worker is an industry-specific exogenous shock with a constant expected value and a constant variance per unit of time. Intermediate goods are imperfect substitutes in the production of a final consumption good, so industry demand curves slope down, although all workers and firms are price-takers.

Workers have standard preferences over consumption and leisure, and markets are complete. At any point in time, workers engage in four mutually exclusive activities, from which they derive different amounts of leisure: work, search unemployment, rest unemployment, and inactivity, i.e. out of the labor force. A worker in a given industry can either work, engage in rest unemployment, or leave the industry. A rest-unemployed worker is available to return to work in that industry, and that industry only, at no cost. If a worker leaves her industry she can either be inactive or search unemployed. Search-unemployed workers find a new industry after a random, exponentially distributed amount of time. We consider both the case of directed search, as in [Lucas and Prescott \(1974\)](#), where a successful searcher moves to the industry of her choice, and of random search, where the probability that a successful searcher moves to any particular industry is proportional to the number of workers in that industry, in the spirit of the [McCall \(1970\)](#) search model. Finally, workers can costlessly move between search unemployment and inactivity.

We perform a quantitative evaluation of the model, focusing on the behavior of industry wages, the key equilibrating force in the [Lucas and Prescott \(1974\)](#) model. Idiosyncratic productivity shocks cause wage dispersion across industries, but we show analytically how this is moderated by workers’ search for better job opportunities. This motivates our empirical approach to evaluating the model: we compare the behavior of wages at the industry level

in the model to the U.S. economy, and show a tight theoretical link between the search unemployment rate and the autocorrelation and variance of wages. We find that our model is consistent with the wage data only if the search unemployment rate is low, approximately one quarter of total unemployment; the remaining unemployment is accounted for by rest. We discuss below whether this is empirically reasonable.

1.1 Employment, Unemployment, and Wage Dynamics

To better understand the mechanics of our model, let ω denote the log of the wage that would prevail in a particular industry if all its workers were employed, measured in units of the consumption good; the actual log wage exceeds ω if there is rest unemployment in the industry. With our functional forms, ω is a linear function of log productivity, which by assumption follows a Brownian motion. It is also a linearly decreasing function of the log of the number of workers in the industry, reflecting that an increase in output reduces the price of the industry's good.

We determine the stochastic process for ω using two equilibrium conditions. First, the stochastic process is affected by individual workers' decisions to enter and exit the industry. In particular, some workers may exit to search for another industry, raising ω . Conversely, the arrival of successful searchers puts downward pressure on ω in a way that depends on whether search is directed or random. Second, the decision of a worker to enter or exit an industry depends on the current value of ω and on its stochastic process. Workers exit an industry if current and future wages are expected to be low. In the directed search model, successful searchers enter industries with high current and future wages, while in the random search model they enter all industries. These conditions define a fixed point problem for the stochastic process. We prove there is a unique fixed point and that ω follows a regulated Brownian motion with endogenous barriers and drift. We also characterize the barriers and drift as functions of preferences and technology, including whether search is random or directed.

Given any equilibrium stochastic process for ω , we can compute the employment rate and the search and rest unemployment rates. First we solve for the unique invariant distribution for ω across workers. In an industry at the lower barrier for ω , a fraction of workers leave and become search unemployed, which determines the incidence of search unemployment. Since the average duration of search unemployment is exogenous, this pins down the search unemployment rate and provides our link between the stochastic process for wages and the search unemployment rate. We also show that the rest unemployment rate in an industry is continuously decreasing in ω up to some threshold; above that threshold there is full employment. Finally, we compute the economy-wide employment and rest unemployment

rates by aggregating across industries using the invariant distribution for ω .

Search and rest unemployment play distinct roles in our model economy. Search unemployment is a high-cost, irreversible decision that enables workers to move to a new industry. This generates “labor hoarding,” in the sense that the marginal product of labor is not equalized across industries. Rest unemployment is a less costly, reversible decision that does not enable workers to move. Effectively, rest unemployment raises the marginal product of labor in depressed industries by giving workers the option to remain in contact with the industry without incurring the disutility of working. In this sense, the possibility of rest unemployment encourages labor hoarding at the industry level, albeit hoarding that takes the form of unemployment.

In deciding whether to search or rest, workers compare the leisure value of the two activities against the promised wage gains. Search may lead to a higher wage in a new industry, while, to the extent that the old industry’s wage can increase in the future, rest may also lead to higher wages.¹ The cost and benefit of search unemployment—spending time finding a new industry—are independent of the conditions in the current industry. In contrast, the benefit of rest unemployment—the possibility of high future wages in the current industry—is greater in a less depressed industry since wages are persistent. It follows that workers exit only the most depressed industries to search for new opportunities, while workers in less depressed industries may be rest unemployed, waiting for local conditions to improve.

1.2 Quantitative Evaluation

Our analytical results emphasize the separation of optimization—the determination of the stochastic process for ω —and aggregation—the determination of unemployment rates associated with the stochastic process. Indeed, we also show that we can reverse this logic. Starting from observed unemployment rates, we can place restrictions on the barriers and drift in ω and so say something about the stochastic process for wages. We can thus analyze aggregation without specifying preferences and technology. And starting from a stochastic process for wages, we can uniquely determine the relative disutility of work, search unemployment, and rest unemployment. We can thus analyze optimization without specifying the allocation of workers across industries. We use the separation between optimization and aggregation to make three related points about the quantitative behavior of our model.

¹In our model, productivity is a Brownian motion. This assumption is important for our closed-form solutions, but in reality there may be transitory productivity shocks or even predictable seasonal cycles. A temporary downturn would create a strong incentive for workers to rest rather than search for a new industry. Our assumption that all productivity shocks are permanent therefore appears to reduce the incidence of rest unemployment, conditional on the other model parameters.

First, our aggregation result establishes a tight theoretical relationship between the incidence of search unemployment and the autocorrelation of wages at the industry level. Since these results do not depend on the full model, they are robust to the specification of preferences and technology. We then test the theoretical relationship using data for five-digit industries in the United States. We find that the annual average of weekly earnings at the industry level is essentially a random walk. According to our model, this implies that wages rarely hit their regulating barriers, and hence they inherit the persistence of productivity, which we assume is a random walk with drift. But since a worker exits an industry only when ω hits its lower barrier, it follows that the incidence of search unemployment in our model is necessarily small. Given the observed duration of unemployment, we conclude that persistent wages are inconsistent with a high search unemployment rate in the United States economy. This finding holds both in the directed and random search models.

Second, we use our optimization results to show that, in order for our model to match the persistence of wages at the industry level, the disutility of search must be significantly greater than the disutility of work. This result again uses the fact that persistent wages rarely hit their regulating barriers. Combining this with evidence on the variance of wage growth, it follows that the difference between the highest and lowest wage is large, and so are the gains from search for a worker in a low-wage industry. Workers' optimization then implies that the cost of search is large as well. This finding is closely related to the conclusion in [Hornstein, Krusell, and Violante \(2009\)](#) that search models with plausible search costs cannot generate much wage dispersion. We conjecture that some of the disutility of search actually stands in for different forms of industry-specific human capital, a topic that we have begun to explore in other papers ([Alvarez and Shimer, 2009a,b](#)).

Third, we use our aggregation results to show that the model can allow for rest unemployment to account for three-quarters of all unemployment, while still having persistent wages. Thus there is no conflict between a high rest unemployment rate and persistent wages. Mechanically, given the barriers and drift in ω , high rest unemployment simply lowers employment and so raises the wage in some depressed industries, with little effect on the measured autocorrelation of wages. The question remains whether it is reasonable to assume that rest unemployment accounts for three-quarters of all unemployment. To answer this, we need to take a stand on the empirical counterpart of rest unemployment. One feature of rest unemployed workers is that they eventually return to work in their old industry. [Murphy and Topel \(1987\)](#) and [Loungani and Rogerson \(1989\)](#) document that this is the case following about three-quarters of all unemployment spells in the U.S.. We view this as suggesting that our model may offer a reasonable description of unemployment and wage dynamics. We discuss the mapping between our model and this type of data further in [Section 6.4](#).

1.3 Relationship to the Literature

There are four significant differences between our model and [Lucas and Prescott \(1974\)](#). First, we introduce rest unemployment to the framework. Second, we make particular assumptions on the stochastic process for productivity which enable us to obtain closed-form solutions. This leads us to evaluate the model’s properties in novel ways; however, we believe our insights, e.g. on the link between search unemployment and the autocorrelation of wages and on the role of rest unemployment, carry over to alternative productivity processes. Third, in [Lucas and Prescott \(1974\)](#), all industries produce a homogeneous good but there are diminishing returns to scale in each industry. In our model, each industry produces a heterogeneous good and has constant returns to scale. We believe our approach is more attractive because the extent of diminishing returns is determined by the elasticity of substitution between goods, which is potentially more easily measurable than the degree of decreasing returns on variable inputs ([Atkeson, Khan, and Ohanian, 1996](#)). Finally, our characterization of the random search model is new.²

Our concept of rest unemployment is closely related to the one [Jovanovic \(1987\)](#), from whom we borrow the term.³ While in both his model and ours search and rest unemployment coexist, the aims of both papers and hence the setup of the models are different. [Jovanovic \(1987\)](#) focuses on the cyclical behavior of unemployment and productivity, and so allows for both idiosyncratic and aggregate productivity shocks. But to be able to analyze the model with aggregate shocks, [Jovanovic \(1987\)](#) assumes that at the end of each period, there is exactly one worker in each location.

[Hamilton \(1988\)](#), [King \(1990\)](#) and [Gouge and King \(1997\)](#) all develop models of rest unemployment in the [Lucas and Prescott \(1974\)](#) framework with directed search. [King \(1990\)](#) focuses on comparative statics of rest and search unemployment. [Hamilton \(1988\)](#) and [Gouge and King \(1997\)](#) reexamine the business cycle issues in [Jovanovic \(1987\)](#), such as the finding that rest unemployment is likely to be countercyclical, consistent with empirical evidence but not with many models of reallocation. The key difference between those papers and ours is the stochastic process for productivity. While we assume productivity follows a geometric Brownian motion, these earlier papers assume a two-state Markov process. This coarse parameterization makes their analysis of cyclical fluctuations tractable, but makes it

²The closest random search model is the one in [Alvarez and Veracierto \(1999\)](#). That paper assumes that each searcher is equally likely to find any industry, while we assume here that searchers are more likely to find industries with more workers, as in [Burdett and Vishwanath \(1988\)](#). We view this assumption as no less plausible and it is much more tractable in our current setup.

³An alternative is “wait unemployment,” but the literature that uses this term emphasizes the behavior of workers waiting for a job in a high wage primary labor market rather than accepting a readily available job in a low wage secondary labor market. We study a related concept in our work on unionization in [Alvarez and Shimer \(2009b\)](#).

harder to map the model into rich cross-sectional data. For example, wages in their models also follow a two-state Markov process. One of our main contributions is to show that when log productivity follows a Brownian motion, the model is still tractable and the mapping from model to cross-sectional data is direct. Of course, solving our version of the model with aggregate shocks remains a daunting task. In this sense, we view the two approaches as complementary.

It is worth stressing that these papers and, to our knowledge, all previous research based on the [Lucas and Prescott \(1974\)](#) search model, assume that productivity is mean-reverting. We conjecture that such an assumption would strengthen our main quantitative finding, that this framework has difficulty generating a near random walk in industry-level wages when the incidence of search unemployment is high. The intuition is straightforward: mean reversion in productivity creates mean reversion in wages even without any endogenous movement of workers in and out of industries.

Our paper is also related to a large literature on how labor hoarding affects measured productivity. A number of authors have argued that labor hoarding increases the measured procyclicality of labor productivity ([Summers, 1986](#); [Burnside, Eichenbaum, and Rebelo, 1993](#)), while others have argued that it may have the opposite effect ([Prescott, 1986](#); [Horning, 1994](#)). Although our model is not about business cycles, it too suggests that labor hoarding may reduce the response of measured productivity to shocks at the industry level. The key insight is that labor is hoarded by industries rather than by firms and so takes the form of unemployment rather than employment. Following an adverse shock to an industry, some workers become rest unemployed, which raises the productivity—output in units of the final good per worker—of the remaining workers.

In [Section 2](#), we describe the economic environment. We analyze a special case where workers can immediately move to the best industry in [Section 3](#). Without any search cost, there is no rest unemployment, since either working in the best industry or dropping out of the labor force dominates this activity. Instead, idiosyncratic productivity shocks lead to a continual reallocation of workers across industries.

[Section 4](#) characterizes the directed search model. We describe the equilibrium as the solution to a system of two equations in two endogenous variables and various model parameters. We prove that the equilibrium is unique and perform simple comparative statics. In particular, we find that there is rest unemployment only if the cost in terms of foregone leisure is low. We also provide closed form expressions for the employment, search unemployment, and rest unemployment rates, as well as aggregate output. [Section 5](#) gives a parallel characterization of the random search model, now describing the equilibrium as two equations in two different endogenous variables. We again provide closed form expressions for

employment, search unemployment, rest unemployment, and output. The arguments are similar to the directed search case and so we keep our presentation compact. [Section 6](#) develops our measure of wage persistence and quantitatively compares our model with U.S. data. [Section 7](#) concludes.

2 Model

We consider a continuous time, infinite-horizon model. We focus for simplicity on an aggregate steady state and assume markets are complete.

2.1 Intermediate Goods

There is a continuum of intermediate goods indexed by $j \in [0, 1]$. Each good is produced in a separate industry with a constant returns to scale technology that uses only labor. In a typical industry j at time t , there is a measure $l(j, t)$ workers. Of these, $e(j, t)$ are employed, each producing $x(j, t)$ units of good j , while the remainder are rest unemployed. $x(j, t)$ is an idiosyncratic shock that follows a geometric random walk,

$$d \log x(j, t) = \mu_x dt + \sigma_x dz(j, t), \quad (1)$$

where μ_x measures the drift of log productivity, $\sigma_x > 0$ measures the standard deviation, and $z(j, t)$ is a standard Wiener process, independent across industries. The price of good j , $p(j, t)$, and the wage in industry j , $w(j, t)$, are determined competitively at each instant t and are expressed in units of the final good.

To keep a well-behaved distribution of labor productivity, we assume that industry j shuts down according to a Poisson process with arrival rate δ , independent across industries and independent of industry j 's productivity. When this shock hits, all the workers are forced out of the industry. A new industry, also named j , enters with positive initial productivity x_0 , keeping the total measure of industries constant. We assume a law of large numbers, so the share of industries experiencing any particular sequence of shocks is deterministic.

2.2 Final Goods

A competitive sector combines the intermediate goods into the final good using a constant returns to scale technology

$$Y(t) = \left(\int_0^1 y(j, t)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \quad (2)$$

where $y(j, t)$ is the input of good j at time t and $\theta > 0$ is the elasticity of substitution across goods. We assume $\theta \neq 1$ throughout the paper and comment in [Section 3](#) on the role of this assumption. The final goods sector takes the price of the intermediate goods $\{p(j, t)\}$ as given and chooses $y(j, t)$ to maximize profits. It follows that

$$y(j, t) = \frac{Y(t)}{p(j, t)^\theta}. \quad (3)$$

2.3 Households

There is a representative household consisting of a measure 1 of members. The large household structure allows for full risk sharing within each household, a standard device for studying complete markets allocations.

At each moment in time t , each member of the representative household engages in one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the industries.
 - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
 - $U_r(t) = L(t) - E(t)$ of these workers are rest unemployed and get leisure b_r .
- $U_s(t)$ household members are search unemployed, looking for a new industry and getting leisure b_s .
- The remaining household members are inactive, getting leisure $b_i > b_s$.

Household members may costlessly switch between employment and rest unemployment and between inactivity and searching; however, they cannot switch industries without going through a spell of search unemployment.

Workers exit their industry for inactivity or search in three circumstances: first, they may do so endogenously at any time at no cost; second, they must do when their industry shuts down, which happens at rate δ ; and third, they must do so when they are hit by an idiosyncratic shock, according to a Poisson process with arrival rate q , independent across individuals and independent of their industry's productivity. We introduce the idiosyncratic "quit" shock q to account for separations that are unrelated to the state of the industry.

A worker in search unemployment finds a job in the industry of her choice according to a Poisson process with arrival rate α . In [Section 5](#), we consider a variant of the model where search-unemployed workers instead find a random industry.

We can represent the household's preferences via the utility function

$$\int_0^{\infty} e^{-\rho t} (u(C(t)) + b_i(1 - E(t) - U_r(t) - U_s(t)) + b_r U_r(t) + b_s U_s(t)) dt, \quad (4)$$

where $\rho > 0$ is the discount rate, u is increasing, differentiable, strictly concave, and satisfies the Inada conditions $u'(0) = \infty$ and $\lim_{C \rightarrow \infty} u'(C) = 0$, and $C(t)$ is the household's consumption of the final good. The household finances its consumption using its labor income.

2.4 Equilibrium

We look for a competitive equilibrium of this economy. At each instant, each household chooses how much to consume and how to allocate its members between employment in each industry, rest unemployment in each industry, search unemployment, and inactivity, in order to maximize utility subject to technological constraints on reallocating members across industries, taking as given the stochastic process for wages in each industry; each final goods producer maximizes profits by choosing inputs taking as given the price for all the intermediate goods; and each intermediate goods producer j maximizes profits by choosing how many workers to hire taking as given the wage in its industry and the price of its good. Moreover, the demand for labor from intermediate goods producers is equal to the supply from households in each industry; the demand for intermediate goods from the final goods producers is equal to the supply from intermediate goods producers; and the demand for final goods from the households is equal to the supply from the final goods producers.

Standard arguments imply that for given initial conditions, there is at most one competitive equilibrium of this economy.⁴ We look for a stationary equilibrium where all aggregate quantities and the joint distribution of wages, productivity, output, employment, and rest unemployment across industries are constant.

3 Frictionless Model

To understand the mechanics of the model, we start with a version where nonworkers can instantaneously become workers; formally, this is equivalent to the limit of the model when $\alpha \rightarrow \infty$. In this limit, the household does not need to devote any workers to search unemployment. Moreover, assuming $b_i > b_r$, there is no rest unemployment, since resting

⁴The first welfare theorem implies that any equilibrium is Pareto optimal. Since all households have the same preferences and endowments, if there were multiple equilibria, household utility must be equal in each. But a convex combination of the equilibrium allocations would be feasible and Pareto superior, a contradiction.

is dominated by inactivity. Thus the household divides its time between employment and inactivity. Finally, with costless mobility all workers must earn a common wage $w(t)$.

The household therefore solves

$$\max_{E(t)} \int_0^\infty e^{-\rho t} (u(C(t)) + b_i(1 - E(t))) dt,$$

subject to the budget constraint $C(t) = w(t)E(t)$. Assuming an interior equilibrium, $E(t) \in (0, 1)$, the first order conditions imply that at each date t , $b_i = w(t)u'(C(t))$. Put differently, let $\omega(t) \equiv \log w(t) + \log u'(C(t))$ denote the log wage, measured in units of marginal utility. This is pinned down by preferences alone, $\omega(t) = \log b_i$ for all t .

To determine the level of employment and consumption, we aggregate across industries. Consider industry j with productivity $x(j, t)$ and $l(j, t)$ workers at time t . Output is $y(j, t) = l(j, t)x(j, t)$ and so [equation \(3\)](#) implies the price of the good is $p(j, t) = (Y(t)/l(j, t)x(j, t))^{1/\theta}$. Since workers are paid their marginal revenue product, $p(j, t)x(j, t)$, the log wage measured in units of marginal utility is

$$\omega(t) = \frac{\log Y(t) + (\theta - 1) \log x(j, t) - \log l(j, t)}{\theta} + \log u'(C(t)). \quad (5)$$

We showed in the previous paragraph that $\omega(t)$ is pinned down by preferences, while consumption and output are aggregate variables and so are the same in different industries. Thus this equation determines how employment moves with productivity. When $\theta > 1$, more productive industries employ more workers, while if goods are poor substitutes, $\theta < 1$, an increase in labor productivity lowers employment so as to keep output relatively constant. If $\theta = 1$, employment is independent of productivity, an uninteresting case whose we omit its analysis from the rest of the paper.

To close the model, eliminate $l(j, t)$ from $y(j, t) = x(j, t)l(j, t)$ using [equation \(5\)](#). This gives $y(j, t) = Y(t)(x(j, t)u'(C(t))e^{-\omega(t)})^\theta$. Substitute this into [equation \(2\)](#) and simplify to obtain

$$\frac{e^{\omega(t)}}{u'(C(t))} = \left(\int_0^1 x(j, t)^{\theta-1} dj \right)^{\frac{1}{\theta-1}}.$$

With an invariant distribution for x , we can rewrite this equation by integrating across that distribution. [Appendix A.1](#) finds an expression for the invariant distribution. Assuming

$$\delta > (\theta - 1) \left(\mu_x + (\theta - 1) \frac{\sigma_x^2}{2} \right), \quad (6)$$

we prove that

$$\frac{e^{\omega(t)}}{u'(C(t))} = x_0 \left(\frac{\delta}{\delta - (\theta - 1) \left(\mu_x + (\theta - 1) \frac{\sigma_x^2}{2} \right)} \right)^{\frac{1}{\theta-1}}. \quad (7)$$

This equation pins down aggregate consumption C and hence aggregate output Y in terms of model parameters. If condition (6) fails with $\theta > 1$, extremely productive firms would produce an enormous amount of easily-substitutable goods, driving consumption to infinity. If it fails with $\theta < 1$, extremely unproductive firms would require a huge amount of labor to produce any of the poorly-substitutable goods, driving consumption to zero.

Since $e^{\omega(t)} = b_i$, equation (7) determines the constant level of consumption $C = C(t)$. Finally, employment is also constant and determined from the budget constraint as $E = Cu'(C)e^{-\omega}$.

For future reference, we note two important properties of the frictionless economy, both of which carry over to the frictional economy. First, if $\mu_x + \frac{1}{2}(\theta - 1)\sigma_x^2 = 0$, $w = x_0$, independent of δ . The condition is equivalent to imposing that $x^{\theta-1}$ is a martingale which, by equation (5), implies employment l is a martingale. This is a reasonable benchmark and ensures that aggregate quantities are well-behaved in the limit as δ converges to zero. Second, an increase in the leisure value of inactivity b_i raises the marginal utility of consumption $u'(C)$ by the same proportion, while employment E decreases in proportion to C .

4 Directed Search Model

We now return to the model where it takes time to find a new industry, $\alpha < \infty$. We look for a steady state equilibrium where the household maintains constant consumption, obtains a constant income stream, and keeps a positive and constant fraction of its workers in each of the activities, employment, rest unemployment, search unemployment, and inactivity.

Equilibrium is a natural generalization of the frictionless model. Because it is costly to switch industries, different industries pay different wages at each point in time. Workers exit industries when wages fall too low, which puts a lower bound on wages. Searchers enter industries when wages rise too high, providing an upper bound. Between these bounds, there is no endogenous entry or exit of workers, and so wages fluctuate only due to exogenous quits and to productivity shocks.

Our approach to characterizing equilibrium mimics our approach in the previous section. Using the household optimization problem alone, we determine the stochastic process for the log wage measured in units of marginal utility. We then turn to aggregation to pin down output and employment.

4.1 Household Optimization

We assume parameter values are such that some household members work, some search, and some are inactive. The household values its members according to the expected present value of marginal utility that they generate either from leisure or from income; this lies in an interval $[\underline{v}, \bar{v}]$. To find these bounds, first consider an individual who is permanently inactive. It is immediate from [equation \(4\)](#) that he contributes b_i/ρ to the household. Since the household may freely shift workers into inactivity, this must be the lower bound on marginal utility,

$$\underline{v} = b_i/\rho. \quad (8)$$

The household can also freely shift workers into search unemployment, so \underline{v} is also the expected present value of marginal utility for a searcher. But compared to an inactive worker, a searcher get low marginal utility today in return for the possibility of moving to an industry where she generates the highest marginal utility. More precisely, a searcher gets flow utility b_s and finds the best industry at rate α , giving a capital gain $\bar{v} - \underline{v}$. Since the present value of her utility is \underline{v} , this implies $\rho \underline{v} = b_s + \alpha(\bar{v} - \underline{v})$, pinning down \bar{v} :

$$\bar{v} = b_i \left(\frac{1}{\rho} + \kappa \right), \text{ where } \kappa \equiv \frac{b_i - b_s}{b_i \alpha} \quad (9)$$

is a measure of search costs, the percentage loss in current utility from searching rather than inactivity times the expected duration of search unemployment $1/\alpha$.

We turn next to the behavior of wages. As in the frictionless model, consider a typical industry j at time t with productivity $x(j, t)$ and $l(j, t)$ workers. If all the workers were employed, labor demand would pin down the log wage measured in units of marginal utility,

$$\omega(j, t) = \frac{\log Y + (\theta - 1) \log x(j, t) - \log l(j, t)}{\theta} + \log u'(C). \quad (10)$$

We omit the derivation of this equation, which is unchanged from [equation \(5\)](#). But if $\omega(j, t) < \log b_r$, the household get more utility from rest unemployment. In equilibrium, employment falls, raising the wage until it is equal to b_r and households are indifferent between the two activities. To stress this point, we call $\omega(j, t)$ the *log full-employment wage* and note that the actual log wage is the maximum of this and $\log b_r$.

We claim that in equilibrium, the expected present value of a household member in an industry depends only on the current log full employment wage,

$$v(\omega) = \mathbb{E} \left(\int_0^\infty e^{-(\rho+q+\delta)t} (\max\{b_r, e^{\omega(j,t)}\} + (q + \delta)\underline{v}) dt \mid \omega(j, 0) = \omega \right), \quad (11)$$

where expectations are taken with respect to future values of the random variable $\omega(j, t)$, whose behavior we discuss further below. The discount rate $\rho + q + \delta$ accounts for impatience, for the possibility that the worker exits the industry exogenously, and for the possibility that the industry ends exogenously. The time- t payoff is the prevailing wage; this holds whether the worker is employed or rest unemployed because when there is rest unemployment, the worker is indifferent between the two states. In addition, if the worker exogenously leaves the industry, which happens with hazard rate $q + \delta$, the household gets a terminal value \underline{v} .

In equilibrium, the expected present value of a worker in an industry must lie in the interval $[\underline{v}, \bar{v}]$. If $v(\omega) > \bar{v}$, searchers would enter the industry, reducing ω and with it $v(\omega)$. If $v(\omega) < \underline{v}$, workers would exit the industry, raising ω and $v(\omega)$.⁵ Thus workers' entry and exit decisions determine two thresholds for the log full employment wage, $\underline{\omega} < \bar{\omega}$, with the value function satisfying

$$v(\omega) \in [\underline{v}, \bar{v}] \text{ for all } \omega, \quad v(\bar{\omega}) = \bar{v}, \text{ and } v(\underline{\omega}) = \underline{v} \text{ if } \underline{\omega} > -\infty. \quad (12)$$

The last equation allows for the possibility that workers never choose to endogenously exit an industry, as will be the case if the leisure from rest unemployment exceeds the leisure from inactivity, $b_r \geq b_i$.

Finally we turn to the dynamics of the log full employment wage, a regulated Brownian motion on the interval $[\underline{\omega}, \bar{\omega}]$. When $\omega(j, t) \in (\underline{\omega}, \bar{\omega})$, only productivity shocks and the deterministic exit of workers change ω . Employment and wages depend on whether there is rest unemployment. Consider the case with $\hat{\omega} > \underline{\omega}$, so there is rest unemployment in the most depressed industries. If $\omega \in [\hat{\omega}, \bar{\omega})$, log employment falls at rate q , there is no rest unemployment, and wages change randomly with productivity. If $\omega \in (\underline{\omega}, \hat{\omega})$, there is rest unemployment, log wages are constant at $\hat{\omega}$, and employment and rest unemployment change randomly with productivity. From [equation \(10\)](#), this implies $d\omega(j, t) = \mu dt + \sigma_\omega dz(j, t)$, where

$$\mu \equiv \frac{\theta - 1}{\theta} \mu_x + \frac{q}{\theta}, \quad \sigma_\omega \equiv \frac{\theta - 1}{\theta} \sigma_x, \quad \text{and } \sigma \equiv |\sigma_\omega|, \quad (13)$$

i.e., in this range $\omega(j, t)$ has drift μ and instantaneous standard deviation σ . When the thresholds $\underline{\omega}$ and $\bar{\omega}$ are finite, they act as reflecting barriers, since productivity shocks that would move ω outside the boundaries are offset by the entry and exit of workers. Expectations in [equation \(11\)](#) are taken with respect to the stochastic process for full employment wages.

These equations uniquely determine the thresholds $\underline{\omega}$ and $\bar{\omega}$:

⁵Note that [equation \(11\)](#) assumes that a worker never endogenously chooses to exit a industry. This is appropriate because a worker never strictly prefers to do so, but rather is always willing to stay if enough other workers exit.

PROPOSITION 1. Equations (11) and (12) uniquely define $\underline{\omega}$ and $\bar{\omega}$ as functions of model parameters. A proportional increase in b_i , b_r , and b_s raises $e^{\underline{\omega}}$ and $e^{\bar{\omega}}$ by the same proportion. Moreover, $\underline{\omega} < \log b_i < \bar{\omega} < \infty$, with $\underline{\omega} > -\infty$ if and only if $b_r < b_i$.

This generalizes the frictionless equilibrium, where household optimization pins down $\omega = \log b_i$. With frictions, the log full-employment wage can fluctuate within some boundaries, with an unchanged relationship between those boundaries are the preference for leisure. There are two ways to prove uniqueness of these boundaries. The proof in [Appendix A.2](#) relies on monotonicity of the value function in the boundaries, showing that an increase in either boundary raises the value function, but an increase in the upper (lower) bound affects v more when ω is close to the upper(lower) bound. Alternatively, in online [Appendix B.2](#) we prove this result using the solution to an “island planner’s problem” (see also [Leahy, 1993](#)).

Notice that the state of an industry in our model is the one dimensional object ω , while in [Lucas and Prescott \(1974\)](#) the state is two dimensional.⁶ [Lucas and Prescott](#) must include productivity x and employment l as separate state variables because they consider a general class of processes for x , in particular allowing for mean reversion. While ω still determines current wages in their setup, it is not a Markov process, so there is no analog of [equation \(13\)](#) in their setup. The assumption that $\log x$ is a Brownian motion with drift permits us to reduce the state variable to a single dimension. This simplification is common in the partial equilibrium literature on irreversible investment ([Bentolila and Bertola, 1990](#); [Abel and Eberly, 1996](#); [Caballero and Engel, 1999](#)) and enables us to provide a more complete analytical characterization of the equilibrium than could [Lucas and Prescott](#).

The Proposition establishes that $\underline{\omega}$ is finite when $b_r < b_i$. Intuitively, if an industry is hit by sufficiently adverse shocks, workers will leave since rest unemployment is costly and has low expected payoffs. In contrast, when $b_r \geq b_i$, rest unemployment is costless and hence workers only leave industries when they shut down. Moreover, if $b_r \leq b_i$, there is no rest unemployment in the best industries, $\bar{\omega} > \log b_r$. The next proposition addresses whether there is rest unemployment in the worst industries, $\underline{\omega} \geq \log b_r$.

PROPOSITION 2. There exists a \bar{b}_r such that in an equilibrium, $b_r \geq e^{\underline{\omega}}$ if and only if $b_r \geq \bar{b}_r$, with $\bar{b}_r = B(\kappa, \rho + q + \delta, \mu, \sigma)b_i$ for some function B , positive-valued and decreasing in κ with $B(0, \rho + q + \delta, \mu, \sigma) = 1$.

The proof is in [Appendix A.3](#). This Proposition implies that there is rest unemployment if search costs κ are sufficiently high given any $b_r > 0$, or equivalently if the leisure value of

⁶ $\max\{b_r, e^{\omega}\}$ is analogous to $R(x, l)$ in [Lucas and Prescott’s \(1974\)](#) notation. Their production technology implies that Y does not affect R , while risk-neutrality ensures that $u'(C)$ is constant. [Lucas and Prescott \(1974\)](#) also assume that $R_x > 0$ —see their equation (1)—which in our set-up is equivalent to $\theta > 1$.

resting b_r is sufficiently close to the leisure of inactivity b_i given any $\kappa > 0$. If a searcher finds a job sufficiently fast (so κ is small) or resting gives too little leisure (so b_r is small), there is no reason to wait for industry conditions to improve, and so B is monotone in κ .

4.2 Aggregation

Having pinned down the thresholds $\underline{\omega}$ and $\bar{\omega}$, we now prove that consumption, employment, and unemployment are uniquely determined. The approach is again analogous to our analysis of the frictionless model.

PROPOSITION 3. There exists a unique equilibrium. The steady state density of workers across log full employment wages is

$$f(\omega) = \frac{\sum_{i=1}^2 |\eta_i + \theta| e^{\eta_i(\omega - \underline{\omega})}}{\sum_{i=1}^2 |\eta_i + \theta| \frac{e^{\eta_i(\bar{\omega} - \underline{\omega})} - 1}{\eta_i}}, \quad (14)$$

where $\eta_1 < 0 < \eta_2$ solve $q + \delta = -\mu\eta + \frac{\sigma^2}{2}\eta^2$.

The proof in [Appendix A.4](#) also provides explicit equations for output and consumption (which are equal by market clearing) and the number of workers in industry L .

We turn next to the rest unemployment rate. Recall that U_r is the fraction of household members who are rest unemployed. If $\log b_r \leq \underline{\omega}$, this is zero. Otherwise, in an industry with $\omega \in [\underline{\omega}, \hat{\omega}]$, the rest unemployment rate is $1 - e^{\theta(\omega - \hat{\omega})}$. Integrating across such industries using [equation \(14\)](#) gives

$$\frac{U_r}{L} = \int_{\underline{\omega}}^{\hat{\omega}} (1 - e^{\theta(\omega - \hat{\omega})}) f(\omega) d\omega = \theta \frac{\frac{e^{\eta_2(\hat{\omega} - \underline{\omega})} - 1}{\eta_2} - \frac{e^{\eta_1(\hat{\omega} - \underline{\omega})} - 1}{\eta_1}}{\sum_{i=1}^2 |\theta + \eta_i| \left(\frac{e^{\eta_i(\bar{\omega} - \underline{\omega})} - 1}{\eta_i} \right)}. \quad (15)$$

The remaining household members who are in industries are employed, $E = L - U_r$.

Finally we pin down the search unemployment rate. Let N_s be the number of workers among L that leave their industry per unit of time, either because conditions are sufficiently bad or because their industry has exogenously shut down. [Appendix A.6](#) takes limits of the discrete time, discrete state space model to show that this rate is given by

$$N_s = \frac{\theta\sigma^2}{2} f(\underline{\omega})L + (q + \delta)L. \quad (16)$$

The first term gives the fraction of workers who leave their industry to keep ω above $\underline{\omega}$. The second term is the fraction of workers who exogenously leave their industry. In steady state, the fraction of workers who leave industries must balance the fraction of workers who arrive

in industries. The latter is given by the fraction of workers engaged in search unemployment U_s , times the rate at which they arrive to the industry α , so $\alpha U_s = N_s$. Solve [equation \(16\)](#) using [equation \(14\)](#) to obtain an expression for the ratio of search unemployment to workers in industries:

$$\frac{U_s}{L} = \frac{1}{\alpha} \left(\frac{\theta \sigma^2}{2} \frac{\eta_2 - \eta_1}{\sum_{i=1}^2 |\theta + \eta_i| \frac{e^{\eta_i(\bar{\omega} - \underline{\omega})} - 1}{\eta_i}} + q + \delta \right) \quad (17)$$

To have an interior equilibrium we require that $U_s + U_r + E \leq 1$ so that the labor force is smaller than the total population.⁷

We deliberately leave the expressions for unemployment as a function of the thresholds $\underline{\omega}$, $\hat{\omega}$, and $\bar{\omega}$ in order disentangle optimization—the choice of thresholds—from the mechanics of aggregation. This has two advantages. First, we find it useful to exploit this dichotomy in our numerical evaluation of the model in [Section 6](#). Second, the expressions for rest and search unemployment as a function of the thresholds are identical in other variants of the model, including the original [Lucas and Prescott \(1974\)](#) model. For example, suppose the curvature of labor demand comes from diminishing returns at the island level, due to a fixed factor, rather than imperfect substitutability (see [footnote 6](#)). Then the analog of the elasticity of substitution is the reciprocal of the elasticity of revenue with respect to the fixed factor, while the expressions for unemployment are otherwise unchanged.

We close this section by noting some homogeneity properties of employment, rest unemployment, search unemployment, and consumption.

PROPOSITION 4. Let $b_r = \chi \bar{b}_r$, $b_s = \chi \bar{b}_s$, $b_i = \chi \bar{b}_i$ for fixed \bar{b}_r , \bar{b}_s , and \bar{b}_i . The equilibrium value of the unemployment rate $\frac{U_s + U_r}{U_s + U_r + E}$ and the share of rest unemployed $\frac{U_r}{U_r + U_s}$ do not depend on χ , the level of productivity x_0 , or the utility function u , while $u'(Y)$ is proportional to χ/x_0 .

Proof. By inspection, the unemployment rate and share of rest unemployed are functions of the difference in thresholds $\bar{\omega} - \underline{\omega}$ and $\bar{\omega} - \hat{\omega}$ and the parameters α , δ , q , θ , μ (or μ_x), and σ (or σ_x), either directly or indirectly through the roots η_i . From [Proposition 1](#), the thresholds depend on the same parameters and on the discount rate ρ . Next, [Proposition 1](#) shows that $e^{\underline{\omega}}$ and $e^{\bar{\omega}}$ are proportional to χ . Then [equation \(52\)](#) implies $u'(Y)$ inherits the same proportionality. On the other hand, [Proposition 1](#) implies x_0 does not affect any of the thresholds and so [equation \(52\)](#) implies $u'(Y)$ is inversely proportional to x_0 . \square

⁷If this condition fails, all household members participate in the labor market. The equilibrium is equivalent to one with a higher leisure value of inactivity, the value of b_i such that $U_s + U_r + E = 1$. In any case, [Proposition 4](#) implies that for b_r , b_s , and b_i large enough, the equilibrium has $U_s + U_r + E < 1$.

This proposition shows that the unemployment rate and composition of unemployment is determined by the relative advantage of different leisure activities, while output, and hence consumption and employment, depends on an absolute comparison of leisure versus market production. Indeed, the finding that $u'(Y)$ is proportional to χ/x_0 holds in the frictionless benchmark, where an interior solution for the employment rate requires $b_i = u'(Y)w$, while the wage is proportional to x_0 (see [equation 7](#)). Whether higher productivity lowers or raises equilibrium employment depends on whether income or substitution effects dominate in labor supply. With $u(Y) = \log Y$, an increase in productivity raises consumption proportionately without affecting employment or labor force participation.

4.3 The Limiting Economy

We close this section by discussing a useful limit of the model, when the exogenous shut-down rate of industries δ is zero. We introduced the assumption that industries shut down for technical reasons, to ensure an invariant distribution of productivity and employment. Still, with the parameter restriction, $\mu_x + \frac{1}{2}(\theta - 1)\sigma_x^2 = 0$, discussed previously in [Section 3](#), the economy is well behaved even when δ limits to zero. It is clear from [Proposition 1](#) that $\underline{\omega}$ and $\bar{\omega}$ converge nicely for any value of μ_x as long as the discount rate ρ is positive. More problematic is whether aggregate employment, unemployment, and output converge. This section verifies that the same parameter restriction yields a well-behaved limit of the frictional economy.

When $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$ and $\delta \rightarrow 0$, the roots $\eta_1 < 0 < \eta_2$ in [Proposition 3](#) converge to $\eta_1 = -\theta$ and $\eta_2 = 2q/\theta\sigma^2$. Substituting into [equation \(14\)](#), we find

$$f(\omega) = \frac{\eta_2 e^{\eta_2(\omega - \underline{\omega})}}{e^{\eta_2(\bar{\omega} - \underline{\omega})} - 1}.$$

If $q = 0$ as well, this simplifies further to $f(\omega) = 1/(\bar{\omega} - \underline{\omega})$, i.e. f is uniform on its support, while for positive q the density is increasing in ω . We can also take limits of [equations \(51\)](#) and [\(52\)](#) in [Appendix A.4](#) to prove that the number of workers in industries and output have well-behaved limits.

Turning next to unemployment, [equations \(15\)](#) and [\(17\)](#) imply that in the limit,

$$\frac{U_r}{L} = \frac{\theta \frac{e^{\eta_2(\bar{\omega} - \underline{\omega})} - 1}{\eta_2} + e^{-\theta(\bar{\omega} - \underline{\omega})} - 1}{(\theta + \eta_2) \left(\frac{e^{\eta_2(\bar{\omega} - \underline{\omega})} - 1}{\eta_2} \right)} \quad \text{and} \quad \frac{U_s}{L} = \frac{q}{\alpha(1 - e^{-\eta_2(\bar{\omega} - \underline{\omega})})}. \quad (18)$$

Each of these expressions simplifies further when there are no quits, $q = 0$ and so $\eta_2 \rightarrow 0$:⁸

$$\frac{U_r}{L} = \frac{\theta(\hat{\omega} - \underline{\omega}) + e^{-\theta(\hat{\omega} - \underline{\omega})} - 1}{\theta(\bar{\omega} - \underline{\omega})} \quad \text{and} \quad \frac{U_s}{L} = \frac{\theta\sigma^2}{2\alpha(\bar{\omega} - \underline{\omega})}. \quad (19)$$

The last expression is particularly intuitive. With our parameter restrictions, $\mu = -\frac{1}{2}\theta\sigma^2$. Then $-(\bar{\omega} - \underline{\omega})/\mu$ reflects the average duration of an industry spell, since wages must fall from $\bar{\omega}$ to $\underline{\omega}$ and drift down on average at rate μ , while $1/\alpha$ is the average duration of an unemployment spell. The ratio of durations is the L/U_s .

5 Random Search Model

In this section we analyze an alternative technology for search. Instead of locating the best industry, we assume that agents can only locate industries randomly. As in the directed search case, we obtain a separation between optimization and aggregation, with simple expressions for the reduced form expression for the unemployment rates. This shows that our approach is robust to the specification of the mobility technology. Additionally, we find this alternative specification interesting because wages are not regulated from above, and hence they can, in principle, have very different statistical behavior.

5.1 Setup

As much as possible, our setup parallels the one with directed search. We leave our notation unchanged and focus on the differences between the two models.

The search unemployed engage in one of two mutually exclusive activities. First, $U_{s,r}$ search randomly, finding an industry at rate α . The probability of contacting any particular industry j is proportional to the number of workers in that industry at time t , $l(j, t)$, as in [Burdett and Vishwanath \(1988\)](#). The assumption that workers are not more likely find a high wage industry is an extreme alternative to our directed search model, and it is in the spirit of the search problem in [McCall \(1970\)](#). One interpretation is that a worker searching randomly contacts another worker currently in an industry at rate α , and is equally likely to contact any worker, regardless of the state of her industry. Second, $U_{s,n}$ workers search for a new industry, finding one at the same rate α . This ensures that there are some workers who can get a new industry with productivity x_0 off the ground.

⁸The order of convergence of δ and q to zero does not affect these results.

5.2 Household Optimization

The value of permanent inactivity is given by \underline{v} as described in [equation \(8\)](#). In an equilibrium with inactivity and with search, the expected value of a worker concluding either search activity must be \bar{v} , defined in [equation \(9\)](#).

An industry j at time t is characterized by the log full-employment wage $\omega(j, t)$, unchanged from [equation \(10\)](#), with the value of a worker in such an industry still given by [equation \(11\)](#); however, random search affects the stochastic process for wages and thus the value function. As in the directed search model, ω is regulated from below by agents' willingness to exit an industry with bad prospects, but it is not regulated from above because of the absence of directed search. That is,

$$v(\omega) \geq \underline{v} \text{ for all } \omega \text{ and } v(\underline{\omega}) = \underline{v} \text{ if } \underline{\omega} > -\infty. \quad (20)$$

The arrival of random searchers causes gross labor force growth at some endogenous rate s , a key variable in what follows. This implies that ω follows a Brownian motion regulated below at $\underline{\omega}$ with $d\omega(j, t) = \mu dt + \sigma_\omega dz(j, t)$, where

$$\mu \equiv \frac{\theta - 1}{\theta} \mu_x + \frac{q - s}{\theta}, \quad \sigma_\omega \equiv \frac{\theta - 1}{\theta} \sigma_x, \quad \text{and} \quad \sigma \equiv |\sigma_\omega|. \quad (21)$$

Thus the arrival of random searchers puts downward pressure on wages, the opposite effect of exogenous quits. Indeed, it is not surprising that only the difference between these two rates affects the stochastic process for wages. The behavior of wages and employment on $(\underline{\omega}, \infty)$ is the same to the one in the directed search case on $(\underline{\omega}, \bar{\omega})$, except for the drift of ω , and thus depends on whether there is rest unemployment.

Turn next to the value of workers who have successfully completed a search spell,

$$\int_{\underline{\omega}}^{\infty} v(\omega) f(\omega) d\omega = \bar{v} \quad (22)$$

$$v(\omega_0) = \bar{v}. \quad (23)$$

The first equation states that the expected value of a completed random search spell must be \bar{v} , where $f(\omega)$ is the density of log full-employment wages across industries. The second equation states that the value of a completed search for a new industry must be \bar{v} , which pins down ω_0 , the initial wage in new industries. We prove in [Appendix A.7](#) that the density

of log full-employment wages satisfies

$$f(\omega) = \begin{cases} \frac{\left(\eta_1\eta_2 + \frac{2\delta\bar{L}}{\sigma^2L}\right) \left((\eta_2 + \theta)e^{\eta_2(\omega-\underline{\omega})} - (\eta_1 + \theta)e^{\eta_1(\omega-\underline{\omega})}\right)}{\theta(\eta_2 - \eta_1)} & \text{if } \omega \in [\underline{\omega}, \omega_0] \\ \frac{\left(\eta_1\eta_2 + \frac{2\delta\bar{L}}{\sigma^2L}\right) \left((\eta_2 + \theta)e^{\eta_2(\omega-\underline{\omega})} - (\eta_1 + \theta)e^{\eta_1(\omega-\underline{\omega})}\right)}{\theta(\eta_2 - \eta_1)} \\ \quad + \frac{\frac{2\delta\bar{L}}{\sigma^2L} \left(e^{\eta_1(\omega-\omega_0)} - e^{\eta_2(\omega-\omega_0)}\right)}{\eta_2 - \eta_1} & \text{if } \omega > \omega_0, \end{cases} \quad (24)$$

where $\eta_1 < \eta_2 < 0$ solve $q + \delta - s = -\mu\eta + \frac{\sigma^2}{2}\eta^2$ and \bar{L}/L is the number of workers in a new industry relative to the overall number of workers in industries. Note that there is a kink in the density at ω_0 , reflecting the entry of new industries; if $\delta = 0$, the density is smooth.

To close the model, we describe the ratio \bar{L}/L using the invariant distribution \tilde{f} for the process $\{\omega(t), \log l(t)\}$ across industries. Formally, we can first describe $\{\omega(t)\}$ as a one sided reflected diffusion and jump process on $[\underline{\omega}, \infty)$, and then use this process to describe $\{\log l(t)\}$ on the real line. Let $Z(t)$ be a standard Brownian motion, $N(t)$ the counter associated with a homogeneous Poisson process with intensity δ , and $B(t)$ an increasing singular process:

$$\begin{aligned} \omega(0) &= \omega_0, \quad \log l(0) = \log \bar{L}, \\ d\omega(t) &= \mu dt + \sigma dZ(t) + (\omega_0 - \omega(t))dN(t) + dB(t), \\ d \log l(t) &= (s - q)dt + (\log(\bar{L}) - \log l(t))dN(t) - \theta dB(t), \end{aligned} \quad (25)$$

where for all $T > 0$: $\int_0^T I_{\{\omega(t) > \underline{\omega}\}} dB(t) = 0$. In this definition we start each industry at (ω_0, \bar{L}) . When the industry is destroyed, at rate δ per unit of time, we replace it by a new one with the same initial conditions. Given this recurrence, this process has a unique invariant distribution for all $\delta > 0$. We denote this by $\tilde{f}(\omega, l)$.

The distribution \tilde{f} implies a value for \bar{L}/L . To see this notice that [equation \(25\)](#) implies that an increase in $\log \bar{L}$ increases all the realizations of each path of $\{\log l(t)\}$ by the same amount. Alternatively, [equation \(25\)](#) can be rewritten for the process $\log(l(t)/\bar{L})$, which makes no other reference to \bar{L} . To denote the dependence of \tilde{f} on \bar{L} we write $\tilde{f}(\omega, l; \bar{L})$. We have that L is the measure of workers in industries, so that:

$$L = \int_0^\infty \int_{\underline{\omega}}^\infty l \tilde{f}(\omega, l; \bar{L}) d\omega dl = \bar{L} \int_0^\infty \int_{\underline{\omega}}^\infty l \tilde{f}(\omega, l; 1) d\omega dl.$$

Then we can write a condition for the ratio \bar{L}/L as

$$\bar{L}/L = \left(\int_0^\infty \int_{\underline{\omega}}^\infty l \tilde{f}(\omega, l; 1) d\omega dl \right)^{-1}. \quad (26)$$

Although we do not explicitly solve for $\tilde{f}(\omega, l; 1)$, [equation \(25\)](#) implies that it depends only on $\underline{\omega}$, s , and ω_0 .

[Equations \(20\)](#), [\(22\)](#), [\(23\)](#), and [\(26\)](#) determine $\underline{\omega}$, s , ω_0 , and \bar{L}/L in terms of model parameters. Although we do not have a general existence and uniqueness proof for the random search model, analogous to [Proposition 1](#), later in this section we prove existence in the limit as $\delta \rightarrow 0$. We also establish uniqueness for a particular case, without exogenous quits or rest unemployment.

5.3 Aggregation

To close the model, first determine output Y and the number of workers in industries L . We omit the results, which are unchanged from the directed search model in [Appendix A.4](#). We focus instead on the unemployment rates. For the search unemployed, we have

$$U_{s,r}/L = s/\alpha \quad \text{and} \quad U_{s,n}/L = (\bar{L}/L)(\delta/\alpha). \quad (27)$$

These equations balance inflows and outflows from each state. For example, random searchers flow into industries at rate sL and find jobs at rate α , so there must be $U_{s,r} = sL/\alpha$ such workers. For rest unemployment, we simply have

$$U_r/L = \int_{\underline{\omega}}^{\hat{\omega}} (1 - e^{\theta(\omega - \hat{\omega})}) f(\omega) d\omega, \quad (28)$$

where $\hat{\omega} \equiv \min\{\underline{\omega}, \log b_r\}$.

5.4 The Limiting Economy

We focus again on the special case with $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$, so we can take the limit as δ converges to zero. In this case, the first block of equations describing an equilibrium becomes a set of two equations in two unknowns, namely [equations \(20\)](#) and [\(22\)](#) determining $\underline{\omega}$ and s . This is because neither $\bar{\omega}$ nor \bar{L}/L affect the density f in this limit. Moreover, the roots η_i satisfy

$$\eta_1 = -\theta, \quad \eta_2 = -\frac{2(s - q)}{\theta\sigma^2}, \quad (29)$$

so that the density of workers across industries is exponential:

$$f(\omega) = -\eta_2 e^{\eta_2(\omega - \underline{\omega})}. \quad (30)$$

Using this expression for f , [equation \(28\)](#) reduces to

$$\frac{U_r}{L} = 1 + \frac{\eta_1}{\eta_2 - \eta_1} e^{\eta_2(\hat{\omega} - \underline{\omega})} + \frac{\eta_2}{\eta_1 - \eta_2} e^{\eta_1(\hat{\omega} - \underline{\omega})}. \quad (31)$$

Since no unemployment is accounted for by workers searching for new industries in the limiting economy, we use $U_s = U_{s,r}$, unchanged from [equation \(27\)](#), to denote the search unemployment rate. It is straightforward to prove existence of equilibrium in this limiting economy and, under more restrictive conditions, uniqueness of equilibrium:

PROPOSITION 5. *There exists an equilibrium pair $(\underline{\omega}, s)$ for the random search model with $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$ and $\delta \rightarrow 0$. Moreover, $s > q + \frac{1}{2}\theta\sigma^2$. If $q = b_r = 0$, the equilibrium is unique. In that case, an increase in search costs κ reduces the equilibrium values of s and $\underline{\omega}$, raising the search unemployment rate.*

The proof is in [Appendix A.8](#). We believe the uniqueness result should hold more generally.

6 Quantitative Evaluation

6.1 Auxiliary Statistical Model

We start our quantitative evaluation of the model by describing an auxiliary statistical model for wages in industry j at time t . Suppose that the log wage follows a first order autoregressive process with a common autocorrelation and variance across industries, but possibly with different means for each industry:

$$\log w_{j,t+1} = \beta_w \log w_{j,t} + \log w_{j,\cdot} + \epsilon_{j,t+1}, \quad (32)$$

where $\epsilon_{j,t}$ is normally distributed with mean zero and standard deviation σ_w , and is independent across industries and over time. $w_{j,\cdot}$ represents an industry fixed effect. We can estimate the parameters of the model using exact maximum likelihood; [Appendix A.9](#) provides details on the estimator. Of course, log wages generated by our model do not follow an AR(1), but rather are a regulated Brownian Motion, or an even more nonlinear process in the presence of rest unemployment. Still, we regard [equation \(32\)](#) as useful for summarizing properties of log wages in the data and in the model, in the spirit of indirect inference. The following

proposition explains why:

PROPOSITION 6. Consider a model economy with $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$ and $\delta \rightarrow 0$. Estimate [equation \(32\)](#) using a sample of annual-average wages for J industries during T years. The distribution of the estimated statistics $\hat{\beta}_w$ and $\hat{\sigma}_w/\sigma$ depends only on q , $\theta\sigma$, $\alpha U_s/L$, U_r/L , and on whether search is random or directed.

The proof in [Appendix A.10](#) shows that, after an appropriate normalization, the stochastic process for industry wages depends only on these reduced-form parameters. Note that the Proposition assumes that $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$, which allows us to focus on the limit without exogenous destruction of industries, $\delta \rightarrow 0$. Similar equivalence results hold even with a different drift and with exogenous destruction.

Two comments are in order. First, if there is no rest unemployment and we measure wages at discrete time intervals, i.e. at the end of every year, the infinite-sample value of β_w is a non-monotonic function of $\theta\sigma$, maximized at an interior value; see the proof of [Proposition 6](#) in [Appendix A.10](#). Our numerical simulations suggest that this result carries over to time-aggregated, finite-sample data, with and without rest unemployment. Since one of our main findings is that the model tends to produce too small a value for $\hat{\beta}_w$ relative to the data, we focus our discussion on the value of $\theta\sigma$ that maximizes the expected value (across simulated samples) of $\hat{\beta}_w$ given the other parameters.

Second, in our model economy, there is no reason to include the industry fixed effects $w_{j,\cdot}$ in [equation \(32\)](#). We do so because we are concerned that worker heterogeneity across industries may cause permanent differences in wages. In online [Appendix B.3](#) we offer one justification for this concern, a version of the model with heterogeneity in human capital: most workers are only able to work in a minority of industries, while a few workers can work in all industries. Such scarce labor earns a higher average wage. We find that does not change the distribution of $\hat{\beta}_w$ or $\hat{\sigma}_w$, but instead loads entirely onto the fixed effects $w_{j,\cdot}$. On the other hand, if we dropped the industry fixed effects, our estimate $\hat{\beta}_w$ would be biased towards 1 in this more general environment.

6.2 Persistence of Industry Wages: Data and Model

We estimate [equation \(32\)](#) using U.S. data for five-digit NAICS industries at different levels of aggregation. We measure average weekly earnings in industry j and year t , $\tilde{w}_{j,t}$, for $J = 297$ industries from 1990 to 2008 using data from the Current Employment Statistics (<http://www.bls.gov/ces/>). This is all the industries and years with available data. Deflate this by average weekly earnings in the private sector, \bar{w}_t , to obtain our empirical measure of

aggregation	J	$\hat{\beta}_w$	$\hat{\sigma}_w$
2 digit	15	0.954	0.014
3 digit	75	0.905	0.025
4 digit	227	0.883	0.030
5 digit	297	0.874	0.037
6 digit	157	0.870	0.036

Table 1: Maximum likelihood estimates of [equation \(32\)](#) using CES data from 1990 to 2008 at different levels of aggregation. J indicates the number of industries used in estimation, $\hat{\beta}_w$ is our estimate of autoregression in wages and $\hat{\sigma}_w$ is our estimate of the standard deviation of wage innovations.

wages in industry j at time t is $w_{j,t} \equiv \tilde{w}_{j,t}/\bar{w}_t$.⁹

[Table 1](#) summarizes our empirical findings at different levels of aggregation. Our main focus is on the five-digit level, where our maximum likelihood estimate of the autoregressive coefficient is $\hat{\beta}_w = 0.87$.¹⁰ This number is slightly larger in more aggregated data. The table also shows that the standard deviation of wage changes is about four percent per year at the five-digit level, and is smaller in more aggregate data, presumably reflecting some correlation in shocks.

To get a sense of the magnitude of our persistence estimate, note that $\hat{\beta}_w$ is a biased estimate of β_w due to the finite sample.¹¹ Using Monte Carlo, we find that if $\beta_w = 0.99$ and we estimate [equation \(32\)](#) using $T + 1 = 19$ years of data and $J = 297$ industries, we would obtain $\hat{\beta}_w \approx 0.86$. We are unlikely to be able reject a unit root using our wage data.

[Blanchard and Katz \(1992\)](#) also report a high persistence in relative wages in manufacturing across US states between 1952-1990. They fail to reject a unit root for 47 out of 51 states (including the District of Columbia). Pooling the 51 states and including fixed effects, they estimate an AR(4) using OLS. The implied half life is 11 years and the implied first order autocorrelation is 0.94.¹² Like our estimates in [Table 1](#), these do not correct for finite sample biases and so likely understate the true persistence of the wage process. The bottom line is that industry wages in the data are nearly a random walk.

⁹In the data, the average real wage fluctuates over time, while according to our model, it is constant. But suppose that labor productivity in industry j at time t is $A(t)x(j, t)$, where $A(t)$ is an aggregate shock. With log utility, $u(C) \equiv \log C$, fluctuations in $A(t)$ cause proportional changes in wages measured in units of goods but affect neither wages measured in units of marginal utility, nor the various thresholds measured in units of marginal utility, nor the search or rest unemployment rates. Deflating wages by \bar{w}_t is therefore a perfect control for aggregate productivity shocks, according to our model.

¹⁰We have also estimated [equation \(32\)](#) using OLS and obtained qualitatively similar results.

¹¹See [Nickell \(1981\)](#) for an exact statement of the bias from estimating [equation \(32\)](#) using OLS when J is large and T is finite. We are unaware of a comparable expression using maximum likelihood.

¹²Based upon the numbers reported in the third column of their [Table 1](#).

We next examine the persistence of wages in our theoretical model, focusing throughout on the case with $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$ and $\delta \rightarrow 0$. Rather than explore different values of the structural parameters, we use [Proposition 6](#) to focus on the reduced-form parameters that affect the persistence of wages. For fixed values of the reduced-form parameters, we simulate a discrete time version of our model to construct a data set of wages. We assume a period length is one day and measure the log of annual average wages for $J = 297$ industries and $T + 1 = 19$ years, the same as in the data.¹³ We then estimate [equation \(32\)](#) using the model-generated data. We repeat this many times to obtain an accurate estimate of $\hat{\sigma}_w$.

First suppose there are no exogenous quits, $q = 0$, and only search unemployment, as is the case if the leisure from rest unemployment is zero, $b_r = 0$. Then wage persistence depends only on $\alpha U_s/L$, $\theta\sigma$, and on whether search is random or directed. We measure $U_s/(U_s + L)$ as the unemployment rate, which averaged 5.5 percent in the United States from 1990 to 2008. The mean duration of an in-progress unemployment spell was 0.31 years, which implies $\alpha = 3.2$ per year.¹⁴ Putting this together, the rate that unemployed workers find jobs per employed worker is $\alpha U_s/L \approx 0.186$ per year. Note that in our steady state model, this is also the incidence of search unemployment among employed workers.

Using this number, we find that in the random search model, $\hat{\beta}_w = 0.66$ when $\theta\sigma = 0.60$, and is smaller for other values of $\theta\sigma$. In the directed search model, we find that the maximum obtainable persistence is $\hat{\beta}_w = 0.58$, obtained when $\theta\sigma = 1.30$. Although there is some variation in $\hat{\beta}_w$ across simulations of the model, its standard deviation is below 0.02—see the notes in [Table 2](#)—which is too small to reconcile the model with the data. In summary, the model cannot simultaneously generate a high incidence of search unemployment and persistent wages. Intuitively, this is because a high incidence of search unemployment implies that industry wages are frequently reflecting off the lower bound as workers exit for unemployment, but this implies that persistence must be low.

Exogenous quits moderate these results because the model needs a lower endogenous incidence of search unemployment. For example, set $q = 0.12$, so about two-thirds of unemployment spells are due to exogenous quits, which are independent of industry conditions. We find that the maximum attainable value of $\hat{\beta}_w$ increases to 0.78 in the random search model and 0.71 in the directed search model, still significantly short of the data. Still higher exogenous quit rates, $q = 0.18$, can potentially reconcile the model with the data in [Table 1](#).

¹³We draw an initial value of ω from the ergodic distribution of ω across industries, say $g(\omega)$. This is not the same as the ergodic distribution of ω across workers, $f(\omega)$, because industries that experience negative productivity shocks shed workers but are no more likely to disappear. In the random search model, we find $g(\omega) = -(2\mu/\sigma^2)e^{(2\mu/\sigma^2)(\omega-\underline{\omega})}$ while f is given in [equation \(30\)](#). Both distributions are exponential, but the mean of g is lower. We obtain a similar result in the directed search model.

¹⁴The empirical duration numbers were constructed by the Bureau of Labor Statistics from the Current Population Survey and may be obtained from <http://www.bls.gov/cps/>.

But we do not find this model very interesting. Unemployed workers find jobs at an exogenous rate and lose them mainly due to exogenous quits; the endogenous incidence of search unemployment is $\alpha U_s/L - q = 0.006$ per year. Wages are barely regulated by the entry and exit of workers, and so the equilibrating mechanism in [Lucas and Prescott \(1974\)](#) is nearly shut down.

Now suppose there is some rest unemployment. In this case, our model generates more persistence in wages for a given quit rate. This happens for two reasons. First, the model does not need to put so many industries near the lower bound $\underline{\omega}$; instead, unemployment can occur at higher values of the log full-employment wage. Second, we measure the actual wage, not the full-employment wage, in the data. Industries with rest unemployment pay a constant wage, which raises the measured persistence $\hat{\beta}_w$.

To evaluate the quantitative importance of these effects, suppose that search accounts for one-quarter of all unemployment, $U_s/(U_s + L) = 0.013$, with the remainder accounted for by rest, $U_r/(U_r + L) = 0.042$. This totals the same 5.5% unemployment rate as before. In [Section 6.4](#) we explain why these numbers may be plausible. We keep the arrival rate of job offers to searchers fixed at $\alpha = 3.2$. [Table 2](#) reports the maximum attainable value of the measured autoregressive coefficient $\hat{\beta}_w$, along with the value of $\theta\sigma$ that achieves this maximum, for different values of the exogenous quit rate q and for both random and directed search and directed search models. Note that the structure of our model imposes $q \leq \alpha U_s/L \approx 0.042$.

Two patterns emerge: higher values of q imply more persistence in wages; and for a given value of q , the random search model can generate slightly more persistent wages than the directed search model. The table also shows that rest unemployment significantly raises the measured persistence of wages. In particular, we match the empirical persistence at the five digit level in the random search model with $q = 0.02$ and in the directed search model with $q = 0.03$, in both cases leaving substantial room for endogenous search unemployment as workers exit industries at $\underline{\omega}$. We conclude that rest unemployment potentially helps to reconcile persistent industry wages with high endogenous unemployment rates.

6.3 Structural Parameters

We now return to our structural model and look at how preference parameters affect the measured persistence of wages. We assume throughout that the search unemployment rate is $U_s/(U_s + L) = 0.013$ and the rest unemployment rate is $U_r/(U_r + L) = 0.042$. The search unemployed find jobs at rate $\alpha = 3.2$. For a given value of the exogenous quit rate q , we set $\theta\sigma$ so as to maximize the autoregressive coefficient in wages, i.e. at the values in [Table 2](#).

	q	$\theta\sigma$	$\hat{\beta}_w$	$\hat{\sigma}_w/\sigma$	θ	b_r/b_i	κ
random search	0	0.29	0.842	0.507	4.0	0.95	2.0
	0.01	0.26	0.856	0.517	3.6	0.95	2.6
	0.02	0.21	0.870	0.530	3.0	0.96	3.2
	0.03	0.15	0.885	0.544	2.2	0.96	5.0
	0.04	0.067	0.903	0.561	1.02	—	∞
directed search	0	0.60	0.822	0.488	7.9	0.95	2.7
	0.01	0.57	0.837	0.492	7.6	0.96	3.1
	0.02	0.54	0.852	0.493	7.2	0.96	3.8
	0.03	0.49	0.868	0.491	6.5	0.97	5.0
	0.04	0.41	0.886	0.429	4.8	0.99	10.1

Table 2: Results from random and directed search models with 1.3% search unemployment, 4.2% rest unemployment, and $\alpha = 3.2$. Each row shows a different value of the exogenous quit rate q (first column) and the value of $\theta\sigma$ (second column) that maximizes the maximum likelihood estimate of the autoregressive coefficient $\hat{\beta}_w$ (third column). The standard deviation of the estimate of $\hat{\beta}_w$ across runs of the models ranges from 0.008 to 0.012, depending on the parameterization. The fourth column shows the maximum likelihood estimate of the standard deviation of the residual in [equation \(32\)](#), which is proportional to σ . The standard deviation of the estimate of $\hat{\sigma}_w$ across runs of the model ranges from 0.012 to 0.017. The fifth column shows the value of θ such that $\hat{\sigma}_w = 0.037$. The last two columns impose $\rho = 0.05$ and show the implied structural parameters b_r/b_i and κ . We assume a period length is one day and measure the log of annual earnings for $J = 297$ industries and $T + 1 = 19$ years.

The remainder of this section explains how we use these facts to uniquely identify all of the structural parameters of our model.

To determine θ and σ separately, we use evidence on the estimated standard deviation of the residual in [equation \(32\)](#). The data in [Table 1](#) indicate that $\hat{\sigma}_w = 0.037$ at the five-digit level. The fourth column of [Table 2](#) shows $\hat{\sigma}_w/\sigma$ using model-generated data, which we use to pin down σ . Then the fifth column of [Table 2](#) shows the implied value of the elasticity of substitution θ , which varies from slightly just over 1 to almost 8, depending on the choice of q and on whether search is random or directed. Evidence in [Broda and Weinstein \(2006\)](#) gives us a sense that the middle of this range may be reasonable. They estimate the elasticity of substitution between goods at the five-digit SITC level using international trade data, and report a median elasticity of 2.8 (see their Table IV).¹⁵ We note that there is no mechanical reason why our model had to deliver an elasticity of substitution in this range.

We now determine the values of other structural parameters, starting with the random search model. [Equation \(27\)](#) implies that search-unemployed workers arrive in industries at rate $s = \alpha U_s/L = 0.042$. Then [equations \(29\) and \(31\)](#) determine $\hat{\omega} - \underline{\omega}$, 0.04 with $q = 0$ and 0.11 with $q = 0.02$. Next, building on our analysis of the value function in [Appendix A.8](#), we use [equations \(8\) and \(20\)](#) to pin down the relative leisure values of rest unemployment and inactivity. With $q = 0$, we find $b_r/b_i = 0.95$, while $q = 0.02$ requires a higher relative value of rest unemployment, $b_r/b_i = 0.96$.¹⁶ More generally, in order to generate rest unemployment, b_r/b_i must exceed 0.90 without exogenous quits and 0.89 with $q = 0.02$. This suggests that, while the rest unemployed must pay some cost to remain in contact with their industry, the cost is small. Rest unemployment and inactivity may look quite similar to an outsider who observes individuals' time use, even though the rest unemployed are much more likely to return to work.

Next we consider the search cost $\kappa = (\bar{v} - \underline{v})/b_i$. We find this by computing the expected value of finding a new industry from [equation \(22\)](#). With $q = 0$, we obtain $\kappa = 2.0$, while $q = 0.02$ requires $\kappa = 3.2$. In words, the expected cost of moving to a new industry is equal to the utility from 2 or 3 years of inactivity. A strong autocorrelation in wages requires that most industries are far from the lower threshold $\underline{\omega}$, but this implies that the wage in an average industry is much higher than the wage in the least productive industry. In order for

¹⁵This elasticity is in line with the one used in much of the literature that quantitatively evaluates the [Lucas and Prescott \(1974\)](#) model. Recall that the analog of θ in a model with diminishing returns at the industry level due to a fixed factor is the reciprocal of the elasticity of revenue with respect to the fixed factor. If the fixed factor is capital, then a capital share of $\frac{1}{3}$ is empirically reasonable. [Alvarez and Veracierto \(1999\)](#) set the elasticity of fixed factor to 0.36, [Alvarez and Veracierto \(2001\)](#) set it to 0.23, and [Kambourov and Manovskii \(2007\)](#) set it to 0.32, in line with values of θ between 2.8 and 4.3.

¹⁶In the random search model with $q = 0.04$, our procedure implies $s < q + \frac{1}{2}\theta\sigma^2$, which is inconsistent with our structural model, since the benefits from search are infinite (see [Proposition 5](#)). We indicate this by setting the search costs to infinity and leaving the value of b_r/b_i blank.

workers to be willing to endure such an industry, the cost of moving must be large.

The determination of the structural parameters in the directed search model is similar. We first use the expression for the search unemployment rate in [equation \(18\)](#) to pin down $\bar{\omega} - \underline{\omega}$, 0.54 with $q = 0$ and 0.77 with $q = 0.03$. We then use the equation for the rest unemployment rate to find $\hat{\omega} - \underline{\omega}$, 0.08 without quits and 0.16 with the high quit rate. Finally, [equation \(12\)](#) gives us the remaining parameters. Without quits, we get $b_r/b_i = 0.95$ and $\kappa = 2.7$. With $q = 0.03$, both numbers are larger, $b_r/b_i = 0.97$ and $\kappa = 5.6$. These search costs are even larger than the ones in the random search model, and the intuition is similar.

We have focused here on the relative leisure values from search unemployment, rest unemployment, and inactivity. To do this, we did not need to take a stand on the utility function. For the model to be consistent with balanced growth, we would additionally require $u(C) \equiv \log(C)$. In this case, labor force participation rates pin down the absolute level of the three leisure parameters.

Finally, it is worth stressing that these high values of search costs κ are a consequence of persistent wages and not due to rest unemployment per se. Suppose we shut down rest unemployment but let $q \rightarrow \alpha U_s/L$ so that wages are nearly a random walk, regardless of whether search is random or directed. In either case, the incentive to search for a job is large and so search costs must be large in equilibrium as well, reaching infinity in the limit when wages are a random walk. For example, with a 5.5 percent search unemployment rate and no rest unemployment, $\alpha U_s/L = 0.186$. Set $q = 0.18$ to generate $\hat{\beta}_w = 0.88$ and $\hat{\sigma}_w/\sigma = 0.74$ in the random search model. These reduced-form parameters imply $\theta = 2.2$, $b_r/b_i = 0.95$, and $\kappa = 2.0$, comparable to values in the full model with rest unemployment.

Our finding that search costs must be high if wages are persistent is closely related to [Hornstein, Krusell, and Violante's \(2009\)](#) finding that when search costs are small, search models cannot generate much wage dispersion. More precisely, in their baseline model, wage offers are random and wages within an employment spell are constant. But they also show that if wages during an employment spell are not persistent, small search costs can be reconciled with a large cross-sectional dispersion in wages.

6.4 Discussion

One of our main quantitative findings, established in [Section 6.2](#), is that our model cannot match the empirical persistence of industry wages unless the endogenous incidence of search unemployment is very low. On the other hand, our model can simultaneously have a high rest unemployment rate and industry wages that are as persistent as in the data. An important, unresolved question is the empirical counterpart of search and rest unemployment. According

to our model, a spell of search unemployment ends when a worker moves to a new industry, while a spell of rest unemployment may end with the worker returning to her old industry. This suggests using evidence on “stayers” and “switchers” developed by [Murphy and Topel \(1987\)](#) from the March Current Population Survey and by [Loungani and Rogerson \(1989\)](#) from the Panel Study of Income Dynamics (PSID). These papers classify unemployed workers as stayers if they remain in the same two-digit industry after an unemployment spell and as switchers otherwise. Despite differences in their data sources and methodology, both papers find that workers switch industries after about one quarter of all unemployment spells. Moreover, [Loungani and Rogerson \(1989\)](#) report that switchers account for about a third of all weeks of unemployment, i.e. they stay unemployed slightly longer than stayers. For this reason, we believe our breakdown of 4.2 percent rest unemployment and 1.3 percent search unemployment may be reasonable.

Still, this evidence is far from definitive. First, it is not clear that “industries” in the model correspond to industries in the data. Our theoretical notion of industries has two defining characteristics. On the one hand, the goods produced within an industry are homogeneous while the goods produced in different industries are heterogeneous, as captured by the elasticity of substitution θ . This is consistent with the notion of an industry. On the other hand, workers are free to move within an industry but not between industries, perhaps because of some specificity of human capital or because of geographic mobility costs. To the extent that human capital is occupation, not industry, specific ([Kambourov and Manovskii, 2007](#)), this suggests that our theoretical construct may be closer to an occupation. In any case, it seems possible that many of the workers classified as “stayers” in [Murphy and Topel \(1987\)](#) and [Loungani and Rogerson \(1989\)](#) did in fact actively search for a new labor market, for example by moving to a new geographic area. If the search unemployment rate is substantially higher than 1.3 percent, then our model will have trouble generating the wage persistence that we observe in the data.

Second, one might be concerned that, given how similar the leisure values of rest and inactivity are, many workers in rest unemployment might in practice be misclassified as out of the labor force. To get a sense of the potential importance of this issue, we consider a broader measure of unemployment, including workers who are “marginally attached” or working “part-time for economic reasons.” According to the Bureau of Labor Statistics, this averaged 3.9 percent higher than the official unemployment rate from 1994 to 2008. We therefore increase the rest unemployment rate from 4.2 percent to 8.1 percent, leave the search unemployment rate unchanged at 1.3 percent, and rerun our simulations. Fortunately our results are largely unaffected. In the random search model with $q = 0.02$, the maximum autocorrelation $\hat{\beta}_w$ rises from 0.870 to 0.873, while the elasticity of substitution θ falls from

3.0 to 2.6, the relative value of rest b_r/b_i rises from 0.96 to 0.97, and the search cost κ rises from 3.2 to 3.7. In the directed search model with $q = 0.03$, the maximum autocorrelation falls from 0.868 to 0.866, the elasticity of substitution falls from 6.5 to 5.4, relative value of rest rises slightly from 0.97 to 0.98, and the search cost increases from 5.0 to 6.5. Thus even a substantial change in the incidence of rest unemployment does not much affect our evaluation of our model’s features and shortcomings. The key is that the incidence of endogenous search unemployment must not be too high.

Third, our decision to focus on data at the five-digit level was driven in large part by availability issues, but there is no particular reason to think it is costless to move within five-digit industries and costly to move between them. If we defined an industry at a more disaggregated level, or perhaps looked at a cross between an industry and a geographic area, we would expect to see more search unemployment, which would reduce the persistence of wages. Fortunately, this may be consistent with the data, since [Table 1](#) suggests that wages are less persistent in more disaggregated data. Moreover, our approach to backing out the structural parameters would conclude that search costs are lower with a more disaggregated definition of an industry, since wages are less persistent. This too seems reasonable. In this sense, our model does not uniquely pin down the extent of search and rest unemployment, but rather points to the importance of modeling both in a common framework.

Our second main quantitative finding, established in [Section 6.3](#), is that in order for our model to match the empirical persistence and variance of industry wage growth, the search cost has to be very large. Introducing other mobility costs, such as industry-specific human capital, may alleviate this issue. In [Alvarez and Shimer \(2009a\)](#), we prove an isomorphism between our directed search model and a model with industry-specific human capital. The key difference is quantitative: the time it takes to accumulate industry-specific human capital is likely much longer than the duration of search unemployment. This enables us to calibrate the model without large search costs.

Understanding the full richness of the data may require moving beyond our dichotomous treatment of search and rest unemployment. Instead, many different levels of aggregation may be relevant in reality. Moving between “nearby” industries is likely to be cheap, but nearby industries may receive correlated shocks. Workers are therefore sometimes willing to pay a high cost, or give up much of their human capital, to move to distant industry. Such a model would lead to a more nuanced view of search intensity, with our notion of “search” and “rest” as two extremes, but would be less tractable and hence less comprehensible than our approach. Still, we believe that the basic issues we have identified in this paper are likely to be important in that more general framework.

7 Concluding Remarks

This paper characterizes the equilibrium of the Lucas-Prescott search model with directed and random search. Our characterization features a separation between optimization and aggregation which keeps our analysis tractable. We illustrate the value of this tractability by examining the link between the incidence of search unemployment and the persistence of wages and using this to calibrate key parameters. Of course, our model has many implications that we have yet explored, for example the link between wages and worker flows.

Although this is the first paper to separate optimization from aggregation in the context of the [Lucas and Prescott \(1974\)](#) search model, others have fruitfully used the same idea in different economic environments. [Caballero and Engel \(1993\)](#) evaluate the magnitude and effect of labor adjustment costs by looking at the statistical behavior of employment at the firm level, assuming that in a frictionless model, employment would follow a random walk. [Doms and Dunne \(1998\)](#) perform a similar exercise for capital adjustment costs and irreversibilities. [Cole and Rogerson \(1999\)](#) examine whether the [Mortensen and Pissarides \(1994\)](#) model is consistent with the behavior of job creation and destruction at business cycle frequencies by using a reduced-form version of the original model. [Hall \(1995\)](#) similarly studies the propagation of one-time shocks in an elaboration of this model through a statistical representation of the labor market as a 19-state Markov process.

In a companion paper ([Alvarez and Shimer, 2009b](#)), we develop a version of our model where unions can keep wages above the market-clearing level and allocate jobs based on seniority. While the optimization problem is more complicated, the same separation between optimization and aggregation applies and the link between the incidence of search unemployment and wages is unchanged. We find that unions always lead to temporary layoffs, where union members wait for industry conditions to improve, a natural example of rest unemployment. We find that the hazard rate of reentering unemployed is decreasing in duration for union members who are rationed out of jobs. Interestingly, [Katz and Meyer \(1990\)](#) and [Starr-McCluer \(1993\)](#) find that the hazard rates of workers on temporary layoff decrease more strongly with unemployment duration, compared to other unemployed workers.

A Appendix

A.1 Density of Productivity x

Let $f_x(\tilde{x})$ denote the steady state density of log productivity, $\tilde{x} \equiv \log x$, across industries. This solves a Kolmogorov forward equation: $\delta f_x(\tilde{x}) = -\mu_x f'_x(\tilde{x}) + \frac{\sigma_x^2}{2} f''_x(\tilde{x})$ at all $\tilde{x} \neq \tilde{x}_0 \equiv$

$\log x_0$. The solution to this equation takes the form

$$f_x(\tilde{x}) = \begin{cases} D_1^1 e^{\tilde{\eta}_1 \tilde{x}} + D_2^1 e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} < \log x_0 \\ D_1^2 e^{\tilde{\eta}_1 \tilde{x}} + D_2^2 e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} > \log x_0, \end{cases}$$

where $\tilde{\eta}_1 < 0 < \tilde{\eta}_2$ are the two real roots of the characteristic equation

$$\delta = -\mu_x \tilde{\eta} + \frac{\sigma_x^2}{2} \tilde{\eta}^2. \quad (33)$$

For this to be a well-defined density, integrating to 1 on $(-\infty, \infty)$, we require that $D_1^1 = D_2^2 = 0$. To pin down the remaining constants, we use two more conditions: the density is continuous at $\tilde{x} = \log x_0$; and it integrates to 1. Imposing these boundary conditions delivers

$$f_x(\tilde{x}) = \begin{cases} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\tilde{\eta}_1 - \tilde{\eta}_2} e^{\tilde{\eta}_2(\tilde{x} - \log x_0)} & \text{if } \tilde{x} < \log x_0 \\ \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\tilde{\eta}_1 - \tilde{\eta}_2} e^{\tilde{\eta}_1(\tilde{x} - \log x_0)} & \text{if } \tilde{x} > \log x_0. \end{cases} \quad (34)$$

With this notation, we obtain

$$\left(\int_0^1 x(j, t)^{\theta-1} dj \right)^{\frac{1}{\theta-1}} = \left(\int_{-\infty}^{\infty} e^{(\theta-1)\tilde{x}} f_x(\tilde{x}) d\tilde{x} \right)^{\frac{1}{\theta-1}}. \quad (35)$$

The interior integral converges if $\tilde{\eta}_1 + \theta - 1 < 0 < \tilde{\eta}_2 + \theta - 1$. The definition of $\tilde{\eta}_i$ in [equation \(33\)](#) implies these inequalities are equivalent to condition [\(6\)](#). With this restriction, we can then simplify [equation \(35\)](#) to obtain [equation \(7\)](#).

A.2 Proof of [Proposition 1](#)

Throughout this proof we express the value of a worker v as a function not only on the current log wage ω but also of the lower and upper bound on wages $\underline{\omega}$ and $\bar{\omega}$, say $v(\omega; \underline{\omega}, \bar{\omega})$. We start with a preliminary result.

LEMMA 1. v is continuous and nondecreasing in ω , $\underline{\omega}$, and $\bar{\omega}$. It is strictly increasing in each argument if $\omega \in (\underline{\omega}, \bar{\omega})$ and $\bar{\omega} > \log b_r$.

Proof. Define

$$\Pi(\omega'; \omega; \underline{\omega}, \bar{\omega}) \equiv \mathbb{E} \left(\int_0^{\infty} e^{-(\rho+q+\delta)t} I_{\omega'}(\omega(j, t)) dt \mid \omega(j, 0) = \omega \right),$$

where $I_{\omega'}(\omega(j, t))$ is an indicator function, equal to 1 if $\omega(j, t) < \omega'$ and equal to zero other-

wise. This *discounted occupancy function* evaluates to zero at $\omega' \leq \underline{\omega}$ and to $\frac{1}{\rho+q+\delta}$ at $\omega' \geq \bar{\omega}$. We use $\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega})$ for the density of ω' or the *discounted local time function*, where the subscript denotes the partial derivative with respect to the first argument. Then switching the order of integration in [equation \(11\)](#), which is permissible since for $-\infty \leq \underline{\omega} \leq \bar{\omega} < \infty$ and $\rho + q + \delta > 0$, the function $\max\{b_r, e^{\omega'}\} + (q + \delta)\underline{v}$ is integrable, we get

$$v(\omega; \underline{\omega}, \bar{\omega}) = \int_{\underline{\omega}}^{\bar{\omega}} \left(\max\{b_r, e^{\omega'}\} + (q + \delta)\underline{v} \right) \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) d\omega'. \quad (36)$$

The value of being in an industry with current log full-employment wage ω is equal to the expected value of future ω' weighted by the appropriate discounted local time function. [Stokey \(2009\)](#) proves in Proposition 10.4 that for all $\omega \in [\underline{\omega}, \bar{\omega}]$,

$$\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) = \begin{cases} \frac{(\zeta_2 e^{\zeta_1 \omega + \zeta_2 \bar{\omega}} - \zeta_1 e^{\zeta_1 \bar{\omega} + \zeta_2 \omega}) (\zeta_2 e^{\zeta_2 (\underline{\omega} - \omega')} - \zeta_1 e^{\zeta_1 (\underline{\omega} - \omega')})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})} & \text{if } \underline{\omega} \leq \omega' < \omega \\ \frac{(\zeta_2 e^{\zeta_1 \omega + \zeta_2 \underline{\omega}} - \zeta_1 e^{\zeta_1 \underline{\omega} + \zeta_2 \omega}) (\zeta_2 e^{\zeta_2 (\bar{\omega} - \omega')} - \zeta_1 e^{\zeta_1 (\bar{\omega} - \omega')})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})} & \text{if } \omega \leq \omega' \leq \bar{\omega}, \end{cases} \quad (37)$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation

$$\rho + q + \delta = \mu\zeta + \frac{\sigma^2}{2}\zeta^2. \quad (38)$$

For $\omega < \underline{\omega}$, $\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) = \Pi_{\omega'}(\omega'; \underline{\omega}; \underline{\omega}, \bar{\omega})$ and for $\omega > \bar{\omega}$, $\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) = \Pi_{\omega'}(\omega'; \bar{\omega}; \underline{\omega}, \bar{\omega})$. That v is continuous follows immediately from [equations \(36\) and \(37\)](#). In particular, the latter equation defines $\Pi_{\omega'}$ as a continuous function.

We next prove that the distribution $\Pi(\cdot; \omega; \underline{\omega}, \bar{\omega})$ is increasing in each of ω , $\underline{\omega}$, and $\bar{\omega}$ in the sense of first order stochastic dominance. This follows from differentiating [equation \(37\)](#) with respect to each variable and using simple algebra. One can verify that an increase in $\underline{\omega}$ strictly increases $\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega})$ for all $\omega' \in (\underline{\omega}, \bar{\omega})$. This therefore strictly reduces $\Pi(\omega'; \omega; \underline{\omega}, \bar{\omega})$ for $\omega' \in (\underline{\omega}, \bar{\omega})$. Similarly, an increase in $\bar{\omega}$ strictly reduces $\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega})$ for all $\omega' \in (\underline{\omega}, \bar{\omega})$, which also strictly reduces $\Pi(\omega'; \omega; \underline{\omega}, \bar{\omega})$ for $\omega' \in (\underline{\omega}, \bar{\omega})$. Finally, an increase in ω when $\omega \in (\underline{\omega}, \bar{\omega})$ reduces $\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega})$ for $\omega' \in (\underline{\omega}, \omega)$ and raises it for $\omega' \in (\omega, \bar{\omega})$. Once again, this implies a stochastic dominating shift in Π .

Since the return function $\max\{b_r, e^{\omega'}\} + (q + \delta)\underline{v}$ is nondecreasing in ω' , weak monotonicity of v in each argument follows immediately from [equation \(36\)](#). In addition, the return function is strictly increasing when $\omega' > \log b_r$, and so we obtain strict monotonicity when the support of the integral includes some $\omega' > \log b_r$, i.e. when $\bar{\omega} > \log b_r$. ■

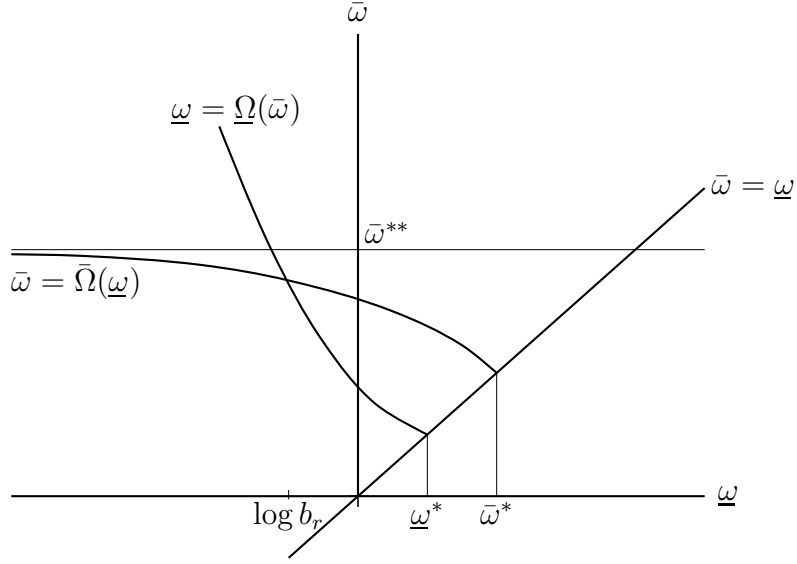


Figure 1: Illustration of the proof of **Proposition 1** when $b_r < b_i$.

Our approach to solving for v may be unfamiliar to some readers, and so it is worth stressing that **equations (12) and (36)** imply some familiar conditions:

$$(\rho + q + \delta)v(\omega; \underline{\omega}, \bar{\omega}) = \max\{b_r, e^\omega\} + (q + \delta)v + \mu v_\omega(\omega; \underline{\omega}, \bar{\omega}) + \frac{\sigma^2}{2}v_{\omega,\omega}(\omega; \underline{\omega}, \bar{\omega}), \quad (39)$$

$$\text{and } v_\omega(\bar{\omega}; \underline{\omega}, \bar{\omega}) = v_\omega(\underline{\omega}; \underline{\omega}, \bar{\omega}) = 0, \quad (40)$$

where subscripts denote partial derivatives with respect to the first argument.¹⁷ Together with the “value-matching” conditions $v(\bar{\omega}; \underline{\omega}, \bar{\omega}) = \bar{v}$ and $v(\underline{\omega}; \underline{\omega}, \bar{\omega}) = \underline{v}$ in **equation (12)**, this is an equivalent representation of the labor force participant’s value function.

We now return to the proof of **Proposition 1**. We start by proving the result when $b_r < b_i$ and defer $b_r \geq b_i$ until the end.

First, define $\bar{\omega}^*$ to solve $v(\bar{\omega}^*; \bar{\omega}^*, \bar{\omega}^*) = \bar{v}$. When ω is regulated at the point $\bar{\omega}^*$, it is trivial to solve **equation (11)** to obtain $\frac{e^{\bar{\omega}^*} + (q + \delta)\underline{v}}{\rho + q + \delta} = \bar{v}$. This point is depicted along the 45° line in **Figure 1**. **Lemma 1** ensures v is continuous and strictly increasing in its first three arguments. Moreover, for any $\underline{\omega} < \bar{\omega}^*$, we can make $v(\bar{\omega}; \underline{\omega}, \bar{\omega})$ unboundedly large by increasing $\bar{\omega}$, while we can make it smaller than \bar{v} by setting $\bar{\omega} = \bar{\omega}^*$. Then by the intermediate value theorem, for any $\underline{\omega} < \bar{\omega}^*$, there exists a $\bar{\Omega}(\underline{\omega}) > \bar{\omega}^*$ solving $v(\bar{\Omega}(\underline{\omega}); \underline{\omega}, \bar{\Omega}(\underline{\omega})) \equiv \bar{v}$. Continuity of v ensures $\bar{\Omega}$ is continuous while monotonicity of v ensures it is decreasing. In addition, because

¹⁷The first condition, the Hamilton-Jacobi-Bellman equation, can be verified directly by differentiating **equation (36)** using the definition of Π_ω in **equation (37)**. The interested reader can consult the online **Appendix B.1** for the details of the algebra. The second pair of conditions, “smooth-pasting,” follow from **equation (36)** because **equation (37)** implies $\frac{\partial \Pi_\omega(\omega'; \omega; \underline{\omega}, \bar{\omega})}{\partial \omega} = 0$ when $\omega = \underline{\omega}$ or $\omega = \bar{\omega}$.

the period return function $s(\omega) \equiv \max\{b_r, e^\omega\} + (q + \delta)v$ is bounded below but not above, $\bar{\omega}^{**} \equiv \lim_{\underline{\omega} \rightarrow -\infty} \bar{\Omega}(\underline{\omega})$ is finite. Thus $\bar{\Omega}(\underline{\omega}) \in (\bar{\omega}^*, \bar{\omega}^{**})$ for any $\underline{\omega} < \bar{\omega}^*$. **Figure 1** illustrates this function.

Similarly, define $\underline{\omega}^*$ to solve $v(\underline{\omega}^*; \underline{\omega}^*, \underline{\omega}^*) = \underline{v}$. Again solve **equation (11)** to obtain $\frac{e^{\underline{\omega}^*} + (q + \delta)v}{\rho + q + \delta} = \underline{v}$. Since $\underline{v} < \bar{v}$, $\underline{\omega}^* < \bar{\omega}^*$, while **equation (8)** implies $\underline{\omega}^* = \log b_i$. For any $\bar{\omega} > \underline{\omega}^*$, we can make $v(\underline{\omega}; \underline{\omega}, \bar{\omega})$ approach $\frac{\rho b_r + (q + \delta)b_i}{\rho(\rho + q + \delta)} < \underline{v}$ by making $\underline{\omega}$ arbitrarily small, while we can make it bigger than \underline{v} by setting $\underline{\omega} = \underline{\omega}^*$. Then by the intermediate value theorem, for any $\bar{\omega} > \underline{\omega}^*$, there exists a $\underline{\Omega}(\bar{\omega}) < \underline{\omega}^*$ solving $v(\underline{\Omega}(\bar{\omega}); \underline{\Omega}(\bar{\omega}), \bar{\omega}) \equiv \underline{v}$. Continuity of v ensures $\underline{\Omega}$ is continuous while monotonicity of v ensures it is decreasing. Thus $\underline{\Omega}(\bar{\omega}) < \underline{\omega}^*$ for any $\bar{\omega} > \underline{\omega}^*$.

An equilibrium is simply a fixed point $\bar{\omega}$ of the composition of the functions $\bar{\Omega} \circ \underline{\Omega}$. The preceding argument implies that this composition maps $[\bar{\omega}^*, \bar{\omega}^{**}]$ into itself and is continuous, and hence has a fixed point.

To prove the uniqueness of the fixed point when $b_r < b_i$, we prove that the composition of the two functions has a slope less than 1, i.e. $\bar{\Omega}'(\underline{\Omega}(\bar{\omega}))\underline{\Omega}'(\bar{\omega}) < 1$. To start, simple transformations of **equation (37)** imply that the cross partial derivatives of the discounted occupancy function satisfy

$$\begin{aligned} \Pi_{\omega, \bar{\omega}}(\omega'; \omega; \underline{\omega}, \bar{\omega}) &= \frac{\zeta_1 \zeta_2 e^{(\zeta_1 + \zeta_2)\bar{\omega}} (e^{-\zeta_1(\omega' - \underline{\omega})} - e^{-\zeta_2(\omega' - \underline{\omega})}) (e^{\zeta_1 \underline{\omega} + \zeta_2 \omega} - e^{\zeta_1 \omega + \zeta_2 \underline{\omega}})}{(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})^2 (\rho + q + \delta)} < 0 \\ \Pi_{\omega, \underline{\omega}}(\omega'; \omega; \underline{\omega}, \bar{\omega}) &= \frac{-\zeta_1 \zeta_2 e^{(\zeta_1 + \zeta_2)\underline{\omega}} (e^{\zeta_2(\bar{\omega} - \omega')} - e^{\zeta_1(\bar{\omega} - \omega')}) (e^{\zeta_1 \omega + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \omega})}{(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})^2 (\rho + q + \delta)} > 0, \end{aligned}$$

where the inequalities use the fact that all the terms in parenthesis are positive. Then use integration-by-parts on **equation (36)** to write

$$v(\omega; \underline{\omega}, \bar{\omega}) = \frac{s(\bar{\omega})}{\rho + q + \delta} - \int_{\underline{\omega}}^{\bar{\omega}} s'(\omega') \Pi(\omega'; \omega; \underline{\omega}, \bar{\omega}) d\omega',$$

where the period return function $s(\omega')$ is nondecreasing and strictly increasing for $\omega' > \log b_r$, and Π is the discounted occupancy function. Taking the cross partial derivatives of this expression gives $v_{\omega, \bar{\omega}}(\omega; \underline{\omega}, \bar{\omega}) > 0 > v_{\omega, \underline{\omega}}(\omega; \underline{\omega}, \bar{\omega})$. In particular, $v_{\bar{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega}) > v_{\bar{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega})$ and $v_{\underline{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega}) > v_{\underline{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega})$. Now since $v_{\omega}(\omega; \underline{\omega}, \bar{\omega}) > 0$ from **Lemma 1**, these inequalities imply

$$\frac{v_{\underline{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega})}{v_{\omega}(\bar{\omega}; \underline{\omega}, \bar{\omega}) + v_{\bar{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega})} \frac{v_{\bar{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega})}{v_{\omega}(\underline{\omega}; \underline{\omega}, \bar{\omega}) + v_{\underline{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega})} < 1.$$

In particular, this is true when evaluated at any point $\{\underline{\omega}, \bar{\omega}\}$ where $\bar{\omega} = \bar{\Omega}(\underline{\omega})$ and $\underline{\omega} = \underline{\Omega}(\bar{\omega})$. Implicit differentiation of the definitions of these functions shows that the first term in the above inequality is $-\bar{\Omega}'(\underline{\omega})$ and the second term is $-\underline{\Omega}'(\bar{\omega})$, which proves $\bar{\Omega}'(\underline{\Omega}(\bar{\omega}))\underline{\Omega}'(\bar{\omega}) < 1$.

Next we prove proportionality of the thresholds $e^{\bar{\omega}}$ and $e^{\underline{\omega}}$ to the leisure values b_r , b_i , and b_s . From [equations \(8\) and \(9\)](#), \underline{v} and \bar{v} are homogeneous of degree one in the three leisure values. The function $\max\{b_r, e^{\omega}\} + (q + \delta)\underline{v}$ is also homogeneous of degree 1 in the leisure values and e^{ω} . By inspection of [equation \(37\)](#), Π_{ω} is unaffected by an equal absolute increase in each of its arguments. Then the integral in [equation \(36\)](#) is homogeneous of degree one in the b 's and $e^{\bar{\omega}}$ and $e^{\underline{\omega}}$. The result follows from [equation \(12\)](#).

Finally we consider $b_r \geq b_i$, so the period return function $s(\omega) \geq b_i + (q + \delta)\underline{v}$ for all ω . This implies $v(\omega; -\infty, \bar{\omega}) \geq \underline{v}$ for all ω and $\bar{\omega}$. Then an equilibrium is defined by $v(\bar{\omega}; -\infty, \bar{\omega}) = \bar{v}$. As discussed above, the solution of this equation is $\bar{\omega}^{**} \in (\bar{\omega}^*, \infty)$.

A.3 Proof of [Proposition 2](#)

First, set $b_r = 0$. By [Proposition 1](#), there exists a unique equilibrium characterized by thresholds $\underline{\omega}_0$ and $\bar{\omega}_0$. We now prove that $\bar{b}_r \equiv e^{\underline{\omega}_0}$. To see why, observe that for all $b_r \leq \bar{b}_r$, the equations characterizing equilibrium are unchanged from the case of $b_r = 0$ because $\log b_r \leq \underline{\omega}_0$, and hence the equilibrium is unchanged. Conversely, for all $b_r > \bar{b}_r$, the equations characterizing equilibrium necessarily are changed, and so the equilibrium must have $\log b_r > \underline{\omega}_{b_r}$.

Next we prove that $\bar{b}_r/b_i = B(\kappa, \rho + q + \delta, \mu, \sigma)$. Again with $b_r = 0$, combine [equations \(12\) and \(36\)](#), noting the discounted local time function Π_{ω} integrates to $\frac{1}{\rho + q + \delta}$, and use the definitions of \underline{v} and \bar{v} in [equations \(8\) and \(9\)](#):

$$\frac{b_i}{\rho + q + \delta} = \int_{\underline{\omega}_0}^{\bar{\omega}_0} e^{\omega} \Pi_{\omega}(\omega; \underline{\omega}_0; \underline{\omega}_0, \bar{\omega}_0) d\omega \quad \text{and} \quad b_i \left(\frac{1}{\rho + q + \delta} + \kappa \right) = \int_{\underline{\omega}_0}^{\bar{\omega}_0} e^{\omega} \Pi_{\omega}(\omega; \bar{\omega}_0; \underline{\omega}_0, \bar{\omega}_0) d\omega.$$

Since Π_{ω} is homogeneous of degree zero in the exponentials of its arguments (see [equation 37](#)), this implies $e^{\underline{\omega}_0}$ and $e^{\bar{\omega}_0}$ are homogeneous of degree 1 in b_i . Moreover, ζ_i depends on $\rho + q + \delta$, μ , and σ by [equation \(38\)](#) and so the density Π_{ω} in [equation \(37\)](#) depends on these same parameters. It follows that the solution to these equations can depend only on these parameters and the parameters on the left hand side of the above equations. In particular, this proves $e^{\underline{\omega}_0} = b_i B(\kappa, \rho + q + \delta, \mu, \sigma)$. Since $\bar{b}_r = e^{\underline{\omega}_0}$, that establishes the dependence of \bar{b}_r on this limited set of parameters.

Obviously B is positive-valued. By [Proposition 1](#), $\underline{\omega}_0 < \log b_i$ and so $B < 1$. We finally prove it is decreasing in κ . Since κ affects $\underline{\omega}$ and $\bar{\omega}$ only through \bar{v} , to establish that B is decreasing in κ it suffices to show that the $\underline{\omega}$ and $\bar{\omega}$ that solve [equations \(12\) and \(36\)](#) is decreasing in \bar{v} . This follows because $\bar{\Omega}(\underline{\omega})$ is increasing in \bar{v} and $\underline{\Omega}(\bar{\omega})$ is unaffected, where these functions are defined in the proof of [Proposition 1](#). A decrease in \bar{v} then reduces the

composition $\bar{\Omega} \circ \underline{\Omega}$. Since the slope of this function is less than 1, it reduces the location of the fixed point $\bar{\omega}$ and hence raises $\underline{\omega} = \underline{\Omega}(\bar{\omega})$.

A.4 Proof of Proposition 3

We start by characterizing the stationary distribution of workers across log full-employment wages ω , f , defined on $[\underline{\omega}, \bar{\omega}]$. Since f is a density,

$$\int_{\underline{\omega}}^{\bar{\omega}} f(\omega) d\omega = 1. \quad (41)$$

By taking the limit of a discrete time, discrete state-space analog of our model, we prove in [Appendix A.5](#) that this density has to satisfy three conditions, [equations \(42\)–\(44\)](#) below. First, in the interior of its support, it must solve a Kolmogorov forward equation,

$$(q + \delta)f(\omega) = -\mu f'(\omega) + \frac{\sigma^2}{2} f''(\omega) \text{ for all } \omega \in (\underline{\omega}, \bar{\omega}). \quad (42)$$

This captures the requirement that inflows and outflows balance at each point in the support of the density. Workers exit industries either because of quits or shutdowns at rate $q + \delta$, while otherwise ω is a Brownian motion with drift μ and standard deviation σ . Workers whose ω changes leave this point in the density for higher or lower values of ω , while the density picks up mass from points above and below when they are hit by appropriate shocks. In continuous time, this relates the level of f and its derivatives.

Second, at the lower bound $\underline{\omega}$,

$$\frac{\sigma^2}{2} f'(\underline{\omega}) - \left(\mu + \frac{\theta \sigma^2}{2} \right) f(\underline{\omega}) = 0. \quad (43)$$

The elasticity of substitution θ appears in this equation because it determines how many workers must exit from depressed industries to regulate ω above $\underline{\omega}$. The exogenous separation rate $q + \delta$ does not appear in this equation because the ratio of endogenous to exogenous exits is infinite in a short time interval for an industry at the lower bound. Since by definition there are no industries with smaller ω , $f(\underline{\omega})$ is not fed from below, which explains the difference between [equations \(42\) and \(43\)](#). The second order differential [equation \(42\)](#) and the boundary conditions [equations \(41\) and \(43\)](#) yield [equation \(14\)](#) using standard calculations.

We now turn to output, consumption (which are equal by the final goods market clearing

condition) and employment. [Appendix A.5](#) also establishes that at the upper bound $\bar{\omega}$,

$$\frac{\sigma^2}{2} f'(\bar{\omega}) - \left(\mu + \frac{\theta\sigma^2}{2} \right) f(\bar{\omega}) = \delta \frac{L_0}{L}, \quad (44)$$

where L_0 is the (endogenous) average number of workers in a new industry. The logic for the left hand side of this equation parallels the logic behind [equation \(43\)](#). There is an extra inflow at $\bar{\omega}$ coming from newly-formed industries, which absorb δL_0 workers per unit of time; dividing by L expresses this as a percentage of the workers located in industries.

Next, [equation \(5\)](#) implies that the number of workers in a new industry is

$$L_0 = Y u'(Y)^\theta x_0^{\theta-1} e^{-\theta\bar{\omega}}, \quad (45)$$

which ensure a log full-employment wage $\bar{\omega}$.

Our last condition relates intermediate and final goods output. Define the productivity of a location x consistent with l workers present in the location, a log full-employment wage ω , and aggregate output and consumption Y . From [equation \(5\)](#), this solves

$$x = \xi(l, \omega, Y) \equiv \left(\frac{l e^{\theta\omega}}{Y u'(Y)^\theta} \right)^{\frac{1}{\theta-1}}. \quad (46)$$

Use this to compute output in an industry with l workers and log full-employment wage ω , recognizing that there may be rest unemployment:

$$Q(l, \xi(l, \omega, Y)) = Y^{\frac{-1}{\theta-1}} \left(\frac{e^{\omega} l}{u'(Y)} \right)^{\frac{\theta}{\theta-1}} \min\{1, e^{\omega}/b_r\}^\theta. \quad (47)$$

Using this notation, we can write [equation \(2\)](#) as

$$\begin{aligned} Y &= \left(\int_0^1 Q(l(j, t), \xi(l(j, t), \omega(j, t), Y))^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \\ &= \left(\int_0^1 Q(L, \xi(L, \omega(j, t), Y))^{\frac{\theta-1}{\theta}} \frac{l(j, t)}{L} dj \right)^{\frac{\theta}{\theta-1}} \end{aligned} \quad (48)$$

Note that these equations have to hold for all t in steady state, so the choice of t is arbitrary. The second equation follows because $Q(\cdot, \xi(\cdot, \omega, Y))^{\frac{\theta-1}{\theta}}$ is linear ([equation 47](#)). To solve this, we change the variable of integration from the name of the industry j to its log full-employment wage ω and number of workers l . Let $\tilde{f}(\omega, l)$ be the ergodic density of the joint distribution of workers in industries (ω, l) . The joint distribution of (ω, l) for an individual worker is a strongly convergent Markov process whenever industries shut down at a positive

rate, $\delta > 0$, which ensures that \tilde{f} is unique. Without characterizing the distribution explicitly, we have $Y = \int_{\underline{\omega}}^{\bar{\omega}} \int_0^\infty Q(L, \xi(L, \omega, Y))^{\frac{\theta-1}{\theta}} \frac{l}{L} \tilde{f}(\omega, l) dl d\omega$. Since $f(\omega) = \int_0^\infty \frac{l}{L} \tilde{f}(\omega, l) dl$, we can solve the inner integral to obtain

$$Y = \left(\int_{\underline{\omega}}^{\bar{\omega}} Q(L, \xi(L, \omega, Y))^{\frac{\theta-1}{\theta}} f(\omega) d\omega \right)^{\frac{\theta}{\theta-1}}. \quad (49)$$

This depends on the known density f rather than the more complicated density \tilde{f} .

Finally, we solve [equations \(44\), \(45\), and \(49\)](#) for L_0 , L , and Y . First eliminate L_0 between [equations \(44\) and \(45\)](#) and evaluate $f(\bar{\omega})$ using [equation \(14\)](#) to get

$$L = \frac{-2\delta Y u'(Y)^\theta x_0^{\theta-1} e^{-\theta\bar{\omega}}}{\sigma^2(\theta + \eta_1)(\theta + \eta_2)(e^{\eta_2(\bar{\omega}-\underline{\omega})} - e^{\eta_1(\bar{\omega}-\underline{\omega})})} \sum_{i=1}^2 |\theta + \eta_i| \frac{e^{\eta_i(\bar{\omega}-\underline{\omega})} - 1}{\eta_i}. \quad (50)$$

Substitute [equation \(47\)](#) into [equation \(49\)](#) to get

$$\begin{aligned} Y &= \frac{L}{u'(Y)} \int_{\underline{\omega}}^{\bar{\omega}} e^\omega \min\{1, e^\omega/b_r\}^{\theta-1} f(\omega) d\omega \\ &= \frac{L e^{\bar{\omega}} \sum_{i=1}^2 |\theta + \eta_i| e^{-(\bar{\omega}-\hat{\omega})} \left(\frac{e^{\eta_i(\hat{\omega}-\underline{\omega})} - e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta + \eta_i} + e^{\eta_i(\bar{\omega}-\underline{\omega})} \frac{e^{(\bar{\omega}-\hat{\omega})} - e^{-\eta_i(\bar{\omega}-\hat{\omega})}}{1 + \eta_i} \right)}{u'(Y) \sum_{i=1}^2 |\theta + \eta_i| \frac{e^{\eta_i(\bar{\omega}-\underline{\omega})} - 1}{\eta_i}}, \end{aligned} \quad (51)$$

where we solve the integral using the expression for f in [equation \(14\)](#) and define $\hat{\omega} \equiv \max\{\underline{\omega}, \log b_r\}$. Eliminating L between these equations and solving for $u'(Y)$ gives an implicit equation for output:

$$\begin{aligned} u'(Y)^{1-\theta} &= \frac{-2\delta x_0^{\theta-1} e^{-(\theta\bar{\omega}-\hat{\omega})}}{\sigma^2(\theta + \eta_1)(\theta + \eta_2)(e^{\eta_2(\bar{\omega}-\underline{\omega})} - e^{\eta_1(\bar{\omega}-\underline{\omega})})} \times \\ &\quad \sum_{i=1}^2 |\theta + \eta_i| \left(\frac{e^{\eta_i(\hat{\omega}-\underline{\omega})} - e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta + \eta_i} + e^{\eta_i(\bar{\omega}-\underline{\omega})} \frac{e^{(\bar{\omega}-\hat{\omega})} - e^{-\eta_i(\bar{\omega}-\hat{\omega})}}{1 + \eta_i} \right). \end{aligned} \quad (52)$$

Finally, [equation \(51\)](#) determines the number of workers in labor markets, L .

A.5 Derivation of the Density f

We use a discrete time, discrete state space model to obtain the Kolmogorov forward equations and boundary conditions for the density f . Divide $[\underline{\omega}, \bar{\omega}]$ into n intervals of length $\Delta\omega = (\bar{\omega} - \underline{\omega})/n$. Let the time period be $\Delta t = (\Delta\omega/\sigma)^2$ and assume that when $\omega < \bar{\omega}$, it decreases with probability $\frac{1}{2}(1 + \Delta p)$ where $\Delta p = \mu\Delta\omega/\sigma^2$; when $\omega > \underline{\omega}$, it increases with

probability $\frac{1}{2}(1 - \Delta p)$; and otherwise ω stays constant. Note that for $\underline{\omega} < \omega(t) < \bar{\omega}$, the expected value of $\omega(t + \Delta t) - \omega(t)$ is $\mu\Delta t$ and the second moment is $\sigma^2\Delta t$. As n goes to infinity, this converges to a regulated Brownian motion with drift μ and standard deviation σ .

Now let $f_n(\omega, t)$ denote the fraction of workers in industries with log full employment wage ω at time t for fixed n . With a slight abuse of notation, let $f_n(\omega)$ be the stationary distribution. We are interested in characterizing the density $f(\omega) = \lim_{n \rightarrow \infty} \frac{f_n(\omega)}{\Delta\omega}$. For $\omega \in [\underline{\omega} + \Delta\omega, \bar{\omega} - \Delta\omega]$, the dynamics of ω imply

$$f_n(\omega, t + \Delta t) = (1 - (q + \delta)\Delta t) \left(\frac{1}{2}(1 + \Delta p)f_n(\omega - \Delta\omega, t) + \frac{1}{2}(1 - \Delta p)f_n(\omega + \Delta\omega, t) \right). \quad (53)$$

In any period of length Δt , a fraction $(q + \delta)\Delta t$ of workers leave due to industry shut downs and idiosyncratic quits. Thus the workers in industries with ω at $t + \Delta t$ are a fraction $1 - (q + \delta)\Delta t$ of those who were in industries at $\omega - \Delta\omega$ at t and had a positive shock, plus the same fraction of those who were in industries at $\omega + \Delta\omega$ at t and had a negative shock. Now impose stationarity on f_n . Take a second order approximation to $f_n(\omega + \Delta\omega)$ and $f_n(\omega - \Delta\omega)$ around ω , substituting Δt and Δp by the expressions above:

$$\begin{aligned} f_n(\omega) &= \left(1 - (q + \delta)\frac{\Delta\omega^2}{\sigma^2} \right) \left(f_n(\omega) - \mu\frac{\Delta\omega^2}{\sigma^2}f_n'(\omega) + \frac{\Delta\omega^2}{2}f_n''(\omega) \right) \\ &\Rightarrow (q + \delta)f_n(\omega) = \left(1 - (q + \delta)\frac{\Delta\omega^2}{\sigma^2} \right) \left(-\mu f_n'(\omega) + \frac{\sigma^2}{2}f_n''(\omega) \right) \end{aligned}$$

Taking the limit as n converges to infinity, $\frac{f_n(\omega)}{\Delta\omega} \rightarrow f(\omega)$ solving [equation \(42\)](#).

Now consider the behavior of f_n at the lower threshold $\underline{\omega}$. A similar logic implies

$$f_n(\underline{\omega}, t + \Delta t) = (1 - (q + \delta)\Delta t)\frac{1}{2}(1 - \Delta p)(f_n(\underline{\omega} + \Delta\omega, t) + f_n(\underline{\omega}, t)(1 - \Delta\tilde{l})).$$

The workers at $\underline{\omega}$ at $t + \Delta t$ either were at $\underline{\omega} + \Delta\omega$ or at $\underline{\omega}$ at t ; in both cases, they had a negative shock. Moreover, in the latter case, a fraction $\Delta\tilde{l} \equiv \theta\Delta\omega$ of the workers exited the industry to keep ω above $\underline{\omega}$. Again impose stationarity but now take a first order approximation to $f_n(\underline{\omega} + \Delta\omega)$ at $\underline{\omega}$; the higher order terms will drop out later in any case. Replacing Δt , Δp , and $\Delta\tilde{l}$ with the expressions described above gives

$$f_n(\underline{\omega}) = \left(1 - (q + \delta)\frac{\Delta\omega^2}{\sigma^2} \right) \left(1 - \frac{\mu\Delta\omega}{\sigma^2} \right) \left(f_n(\underline{\omega}) \left(1 - \frac{\theta\Delta\omega}{2} \right) + \frac{\Delta\omega}{2}f_n'(\underline{\omega}) \right)$$

Again eliminating terms in $f_n(\underline{\omega})$ and taking the limit as $n \rightarrow \infty$, we obtain $\frac{f_n(\underline{\omega})}{\Delta\omega} \rightarrow f(\underline{\omega})$ solving [equation \(43\)](#).

Now consider the behavior of f_n at the upper threshold $\bar{\omega}$:

$$f_n(\bar{\omega}, t + \Delta t) = (1 - (q + \delta)\Delta t)\frac{1}{2}(1 + \Delta p)(f_n(\bar{\omega} - \Delta\omega, t) + f_n(\bar{\omega}, t)(1 + \Delta\tilde{l})) + \delta\Delta t L_0/L.$$

Compared to the equation at the lower threshold, the only significant change is the last term, which reflects the fact that on average a fraction L_0/L workers enter at the upper threshold when a new industry is created. Recall also that industries are destroyed at rate δ per unit of time and hence $\delta\Delta t L_0/L$ is the fraction of workers added to the upper threshold due to newly created industries. Impose stationarity and take limits to get

$$f_n(\bar{\omega}) = \left(1 - (q + \delta)\frac{\Delta\omega^2}{\sigma^2}\right) \left(1 + \frac{\mu\Delta\omega}{\sigma^2}\right) \left(f_n(\bar{\omega}) \left(1 + \frac{\theta\Delta\omega}{2}\right) - \frac{\Delta\omega}{2}f'_n(\bar{\omega})\right) + \delta\frac{\Delta\omega^2}{\sigma^2}L_0/L$$

Eliminate terms in $f_n(\bar{\omega})$ and take the limit as $n \rightarrow \infty$ to obtain $\frac{f_n(\bar{\omega})}{\Delta\omega} \rightarrow f(\bar{\omega})$ solving [equation \(44\)](#).

A.6 Exit Rates from Industries

A worker exits her industry if the log full-employment wage is $\underline{\omega}$ and the industry is hit by an adverse shock, if the industry closes, or if she quits. In the discrete time, discrete state space model, the first event hits a fraction $\frac{1}{2}\Delta(1 - \Delta p)\tilde{l}$ of the workers who survive in an industry with $\omega = \underline{\omega}$, so $N_s\Delta t \equiv (1 - (q + \delta)\Delta t)\frac{1}{2}(1 - \Delta p)\Delta\tilde{l}f_n(\underline{\omega})L + (q + \delta)\Delta t L$. Reexpress $\Delta\omega$, $\Delta\tilde{l}$, and Δp in terms of Δt , take the limit as $n \rightarrow \infty$, and use $\frac{f_n(\omega)}{\Delta\omega} \rightarrow f(\omega)$, to get [equation \(16\)](#).

A.7 Density f : Random Search

Since f is a density,

$$\int_{\underline{\omega}}^{\infty} f(\omega)d\omega = 1. \tag{54}$$

By taking the limit of a discrete time, discrete state-space analog of our model, we find that this density has to satisfy three additional conditions. First, at $\omega > \underline{\omega}$, it must solve a Kolmogorov forward equation, unchanged from [equation \(42\)](#) except for the effect of random search:

$$(q + \delta - s)f(\omega) = -\mu f'(\omega) + \frac{\sigma^2}{2}f''(\omega) \text{ for all } \omega > \underline{\omega}. \tag{55}$$

Second, at the lower bound $\underline{\omega}$, we have the same condition as in the directed search case, [equation \(43\)](#). Finally, in contrast with the directed search case, there is a kink in f at the

point where new industries are created:

$$f'_-(\bar{\omega}) - f'_+(\bar{\omega}) = \frac{2\delta\bar{L}}{\sigma^2 L}. \quad (56)$$

where \bar{L} is the average number of workers in a new industry. The derivation of this condition, which reflects the addition of directed searchers into new industries, is standard and hence omitted. Solving these equations and emphasizing the dependence of f on the boundary $\underline{\omega}$ and the arrival rate of new workers s , we obtain [equation \(24\)](#).

A.8 Proof of [Proposition 5](#)

Express the value of a worker v as a function not only on the current log wage ω but also of the lower bound on wages $\underline{\omega}$ and the arrival rate of search unemployed workers s , $v(\omega; \underline{\omega}, s)$. We start by proving a preliminary result.

LEMMA 2. v is continuous and nondecreasing in $\underline{\omega}$ and continuous and nonincreasing in s . It is strictly monotone in each argument if $\omega > \underline{\omega}$.

Proof. As in the proof of [Lemma 1](#), we can write the value function as

$$v(\omega; \underline{\omega}, s) = \int_{\underline{\omega}}^{\infty} \left(\max\{b_r, e^{\omega'}\} + (q + \delta)v \right) \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, s) d\omega', \quad (57)$$

where again $\Pi_{\omega'}$ is the discounted local time function. Taking limits of [equation \(37\)](#), we obtain

$$\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, s) = \begin{cases} \frac{\zeta_2 e^{\zeta_1(\omega - \underline{\omega})} (\zeta_2 e^{\zeta_2(\underline{\omega} - \omega')} - \zeta_1 e^{\zeta_1(\underline{\omega} - \omega')})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)} & \text{if } \underline{\omega} \leq \omega' < \omega \\ \frac{\zeta_2 e^{\zeta_2(\underline{\omega} - \omega')} (\zeta_2 e^{\zeta_1(\omega - \underline{\omega})} - \zeta_1 e^{\zeta_2(\omega - \underline{\omega})})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)} & \text{if } \omega \leq \omega' \leq \infty, \end{cases} \quad (58)$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation

$$\rho + q + \delta = \mu(s)\zeta + \frac{\sigma^2}{2}\zeta^2, \quad (59)$$

where $\mu(s)$ is a decreasing function from [equation \(21\)](#). The remainder of the proof follows the structure of the proof of [Lemma 1](#) and so is omitted. ■

Now rewrite [equations \(20\)](#) and [\(22\)](#) to define two implicit functions of s :

$$v(\underline{\omega}_1(s); \underline{\omega}_1(s), s) = \underline{v}, \quad (60)$$

$$\int_{\underline{\omega}_2(s)}^{\infty} v(\omega; \underline{\omega}_2(s), s) f(\omega; \underline{\omega}_2(s), s) d\omega = \bar{v}, \quad (61)$$

where our notation also emphasizes the dependence of the cross-sectional density of ω on $\underline{\omega}$ and s . An equilibrium is a solution s to

$$\underline{\omega}_1(s) = \underline{\omega}_2(s). \quad (62)$$

[Lemma 2](#) and [equation \(30\)](#) imply that these are continuous and increasing functions of s . We then argue that

$$\underline{\omega}_2\left(q + \frac{1}{2}\theta\sigma^2\right) = -\infty < \underline{\omega}_1\left(q + \frac{1}{2}\theta\sigma^2\right) \quad \text{and} \quad \lim_{s \rightarrow \infty} \underline{\omega}_2(s) > \lim_{s \rightarrow \infty} \underline{\omega}_1(s). \quad (63)$$

If $s \leq q + \frac{1}{2}\theta\sigma^2$, the integral on the left hand side of [equation \(61\)](#) does not converge for all $\underline{\omega}_2(s)$, while as $s \rightarrow q + \frac{1}{2}\theta\sigma^2$, $\underline{\omega}_2 \rightarrow -\infty$. On the other hand, $\underline{\omega}_1$ is finite for any s . At the other end of the range, as $s \rightarrow \infty$, the density f puts almost all its weight at $\underline{\omega}$. This implies that the left hand side of [equation \(61\)](#) converges to $v(\underline{\omega}_2(\infty); \underline{\omega}_2(\infty), \infty)$. Since $\bar{v} > \underline{v}$, we must have $\underline{\omega}_2(\infty) > \underline{\omega}_1(\infty)$. The existence of $\underline{\omega}$ solving [equation \(62\)](#) then follows from the intermediate value theorem.

For the case where $b_r = 0$, so there is no rest unemployment, and $q = 0$, [equations \(60\)](#) and [\(61\)](#) simplify further. In this case, the left hand sides of both equations are proportional to $e^{\underline{\omega}}$, hence taking the ratio gives

$$\frac{\bar{v}}{\underline{v}} \equiv 1 + \frac{\rho}{\underline{v}}\kappa = \frac{\eta_2(s)(1 + \zeta_1(s) + \eta_2(s))}{(1 + \eta_2(s))(\zeta_1(s) + \eta_2(s))}, \quad (64)$$

where the notation emphasizes that the roots η_i for f and ζ_i given by [equation \(59\)](#) are functions of μ , which depends on s as in [equation \(21\)](#). In a *Mathematica* file available on request, we prove that the right hand side of [equation \(64\)](#) converges to ∞ as s goes to its lower bound, and it converges to 1 as s goes to ∞ . Moreover, a lengthy algebraic argument shows that this ratio is decreasing in s everywhere. Thus, since $\bar{v}/\underline{v} \geq 1$, there is a unique solution s^* , and hence a unique equilibrium. It is immediate that the equilibrium value of s^* is decreasing in the search cost κ , and since $\underline{\omega}_1$ is increasing, the value of $\underline{\omega}$ decreases too.

A.9 Maximum Likelihood Estimator

We estimate [equation \(32\)](#) using exact maximum likelihood. For each industry j , we assume the first observation is drawn from the the ergodic distribution, i.e. $\log w_{j,0}$ is normally distributed with mean $\log w_{j,\cdot}$ and variance $\sigma_w^2/(1 - \beta_w^2)$. Subsequently, for $t = \{0, 1, \dots, T - 1\}$, $\log w_{j,t+1}$ is normally distributed with mean $\beta_w \log w_{j,t} + (1 - \beta_w) \log w_{j,\cdot}$ and variance σ_w^2 , where in our data, $T = 18$. The log likelihood function is therefore

$$-\frac{1}{2\sigma_w^2} \sum_{j=1}^J \left(\sum_{t=1}^T (\log w_{j,t} - \beta_w \log w_{j,t-1} - (1 - \beta_w) \log w_{j,\cdot})^2 + (1 - \beta_w^2)(\log w_{j,0} - \log w_{j,\cdot})^2 \right) + \frac{1}{2}J \log(1 - \beta_w^2) - J(T + 1) \log \sigma_w - \frac{1}{2}J(T + 1) \log(2\pi).$$

Note that OLS treats $w_{j,0}$ as exogenous, whereas maximum likelihood recognizes that it is drawn from the ergodic distribution and so contains information.

It is straightforward to maximize the likelihood function numerically. We start with an initial guess for $\hat{\beta}_w$ and use that to compute the industry fixed effects $\hat{w}_{j,\cdot}$ using each of those first order conditions. We then compute the standard deviation of the innovations $\hat{\sigma}_w$ from its first order condition. Finally, we use the first order condition for $\hat{\beta}_w$ to update our initial guess and iterate until convergence. The last step of the algorithm involves a cubic equation in $\hat{\beta}_w$, but it always has a unique solution on the interval $(-1, 1)$.

A.10 Proof of [Proposition 6](#)

We start with the random search model. Wages are a Brownian motion $d\omega(j, t) = \mu dt + \sigma dz(j, t)$, regulated on $[\underline{\omega}, \infty)$. Using [equation \(21\)](#), $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$ implies $\mu = -\frac{1}{2}\theta\sigma^2 + (q - s)/\theta$. [Equation \(27\)](#) implies that unemployed workers enter industries at rate $s = \alpha U_s/L$. Now define $\tilde{\omega}(j, t) = (\omega(j, t) - \underline{\omega})/\sigma$. The stochastic process for $\tilde{\omega}$ is

$$d\tilde{\omega}(j, t) = - \left(\frac{1}{2}\theta\sigma + \frac{\alpha U_s/L - q}{\theta\sigma} \right) dt + dz(j, t)$$

for $\tilde{\omega}(j, t) \geq 0$. This depends only on $\theta\sigma$, $\alpha U_s/L$, and q . Next, [equation \(31\)](#) implies $(\hat{\omega} - \underline{\omega})/\sigma$ depends only on U_r/L and the same three reduced-form parameters. This implies that the stochastic process for the maximum of $(\omega(j, t) - \underline{\omega})/\sigma$ and $(\hat{\omega} - \underline{\omega})/\sigma$ depends on those four parameters and so must estimates of $\hat{\beta}_w$ and $\hat{\sigma}_w$ using this type of data. Since this is a linear transformation of $\max\{\omega(j, t), \hat{\omega}\}$, it follows that $\hat{\beta}_w$ and $\hat{\sigma}_w/\sigma$ must also depend only on those four parameters.

Note that the theoretical infinite sample correlation between $\tilde{\omega}(j, t)$ and $\tilde{\omega}(j, t + 1)$ de-

creases when the drift in $\tilde{\omega}(j, t)$ becomes more negative, since $\tilde{\omega}$ spends more time bouncing off its lower bound 0. Since the drift is maximized at $\theta\sigma = \sqrt{2(\alpha U_s/L - q)}$, it follows that the infinite-sample, non-time-aggregated, no rest unemployment estimate of $\hat{\beta}_w$ is maximized at this value of $\theta\sigma$ for given $\alpha U_s/L - q$.

In the directed search model, wages are a Brownian motion $d\omega(j, t) = \mu dt + \sigma dz(j, t)$, regulated on $[\underline{\omega}, \bar{\omega}]$. Using [equation \(13\)](#), $\mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2$ implies $\mu = -\frac{1}{2}\theta\sigma^2 + q/\theta$. Again let $\tilde{\omega}(j, t) = (\omega(j, t) - \underline{\omega})/\sigma$. This has the same autoregressive coefficient $\hat{\beta}_w$ as ω , but satisfies a simpler law of motion,

$$d\tilde{\omega}(j, t) = \left(-\frac{1}{2}\theta\sigma + \frac{q}{\theta\sigma}\right) dt + dz(j, t),$$

for $\tilde{\omega}(j, t) \in [0, (\bar{\omega} - \underline{\omega})/\sigma]$. Invert [equation \(18\)](#) to express this interval as

$$\tilde{\omega}(j, t) \in \left[0, -\frac{\theta\sigma}{2q} \log\left(1 - \frac{qL}{\alpha U_s}\right)\right].$$

Once again, this depends only on $\theta\sigma$, $\alpha U_s/L$, and q . The remainder of the proof is unchanged, except that there is no longer a closed-form solution for the value of $\theta\sigma$ that maximizes $\hat{\beta}_w$. When $\theta\sigma$ is close to zero, $\tilde{\omega}$ lies in a small interval and so is nearly serially uncorrelated. When $\theta\sigma$ is large, $\tilde{\omega}$ has a strong negative drift, which eliminates the serial correlation as in the random search model.

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B Additional Appendixes not for Publication

B.1 Derivation Hamilton-Jacobi-Bellman

This appendix proves that if $v(\omega)$ is given by:

$$v(\omega) = \int_{\underline{\omega}}^{\bar{\omega}} R(\omega') \Pi_{\omega'}(\omega'; \omega) d\omega' \quad (65)$$

for an arbitrary continuous function $R(\cdot)$ and where the local time function $\Pi_{\omega'}(\cdot)$ is given as in [Stokey \(2009\)](#) Proposition 10.4:

$$\Pi_{\omega'}(\omega'; \omega) = \begin{cases} \frac{(\zeta_2 e^{\zeta_1 \omega + \zeta_2 \bar{\omega}} - \zeta_1 e^{\zeta_1 \bar{\omega} + \zeta_2 \omega}) (\zeta_2 e^{\zeta_2 (\underline{\omega} - \omega')} - \zeta_1 e^{\zeta_1 (\underline{\omega} - \omega')})}{(\rho + q + \delta)(\zeta_2 - \zeta_1) (e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})} & \text{if } \underline{\omega} \leq \omega' < \omega \\ \frac{(\zeta_2 e^{\zeta_1 \omega + \zeta_2 \underline{\omega}} - \zeta_1 e^{\zeta_1 \underline{\omega} + \zeta_2 \omega}) (\zeta_2 e^{\zeta_2 (\bar{\omega} - \omega')} - \zeta_1 e^{\zeta_1 (\bar{\omega} - \omega')})}{(\rho + q + \delta)(\zeta_2 - \zeta_1) (e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})} & \text{if } \omega \leq \omega' \leq \bar{\omega}, \end{cases} \quad (66)$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation $\rho + q + \delta = \mu\zeta + \frac{\sigma^2}{2}\zeta^2$, then

$$(\rho + q + \delta)v(\omega) = R(\omega) + \mu v'(\omega) + \frac{\sigma^2}{2}v''(\omega).$$

Proof. Differentiating v with respect to ω we get

$$\begin{aligned} v'(\omega) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega') \Pi_{\omega' \omega}(\omega'; \omega) d\omega' \\ v''(\omega) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega') \Pi_{\omega' \omega \omega}(\omega'; \omega) d\omega' + R(\omega) \left(\lim_{\omega' \uparrow \omega} \Pi_{\omega' \omega}(\omega'; \omega) - \lim_{\omega' \downarrow \omega} \Pi_{\omega' \omega}(\omega'; \omega) \right) \end{aligned}$$

where we use that $\Pi_{\omega'}$ is continuous but $\Pi_{\omega' \omega}$ has a jump at $\omega' = \omega$. Then

$$\begin{aligned} &(\rho + q + \delta)v(\omega) - \mu v'(\omega) - \frac{\sigma^2}{2}v''(\omega) \\ &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega) \left((\rho + q + \delta)\Pi_{\omega'}(\omega'; \omega) - \mu\Pi_{\omega' \omega}(\omega'; \omega) - \frac{\sigma^2}{2}\Pi_{\omega' \omega \omega}(\omega'; \omega) \right) d\omega' \\ &\quad - \frac{\sigma^2}{2}R(\omega) \left(\lim_{\omega' \uparrow \omega} \Pi_{\omega' \omega}(\omega'; \omega) - \lim_{\omega' \downarrow \omega} \Pi_{\omega' \omega}(\omega'; \omega) \right). \end{aligned}$$

Using the functional form of $\Pi_{\omega'}$ we have, for $\omega' < \omega$:

$$\Pi_{\omega'}(\omega'; \omega) = e^{\zeta_1 \omega} \tilde{h}_1(\omega') - e^{\zeta_2 \omega} \tilde{h}_2(\omega')$$

where

$$\begin{aligned}\tilde{h}_1(\omega') &= \frac{\zeta_2 e^{\zeta_2 \bar{\omega}} (\zeta_2 e^{\zeta_2 (\underline{\omega} - \omega')} - \zeta_1 e^{\zeta_1 (\underline{\omega} - \omega')})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})} \\ \text{and } \tilde{h}_2(\omega') &= \frac{\zeta_1 e^{\zeta_1 \bar{\omega}} (\zeta_2 e^{\zeta_2 (\underline{\omega} - \omega')} - \zeta_1 e^{\zeta_1 (\underline{\omega} - \omega')})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}})}.\end{aligned}$$

Thus for all $\omega' < \omega$:

$$\begin{aligned}(\rho + q + \delta)\Pi_{\omega'}(\omega'; \omega) - \mu\Pi_{\omega'\omega}(\omega'; \omega) - \frac{\sigma^2}{2}\Pi_{\omega'\omega\omega}(\omega'; \omega) \\ = [(\rho + q + \delta) - \zeta_1\mu - (\zeta_1)^2\frac{\sigma^2}{2}]e^{\zeta_1\omega}\tilde{h}_1(\omega') - [(\rho + q + \delta) - \zeta_2\mu - (\zeta_2)^2\frac{\sigma^2}{2}]e^{\zeta_2\omega}\tilde{h}_2(\omega') = 0\end{aligned}$$

where the last equality follow from the definition of the roots ζ_i . Hence

$$\int_{\underline{\omega}}^{\omega} R(\omega') \left((\rho + q + \delta)\Pi_{\omega'}(\omega'; \omega) - \mu\Pi_{\omega'\omega}(\omega'; \omega) - \frac{\sigma^2}{2}\Pi_{\omega'\omega\omega}(\omega'; \omega) \right) d\omega' = 0.$$

Using a symmetric calculation for $\omega' > \omega$ we have:

$$\int_{\omega}^{\bar{\omega}} R(\omega') \left((\rho + q + \delta)\Pi_{\omega'}(\omega'; \omega) - \mu\Pi_{\omega'\omega}(\omega'; \omega) - \frac{\sigma^2}{2}\Pi_{\omega'\omega\omega}(\omega'; \omega) \right) d\omega' = 0.$$

Next, differentiating $\Pi_{\omega'}(\omega'; \omega)$ when $\omega' < \omega$ and when $\omega' > \omega$ and let $\omega' \rightarrow \omega$ from below and from above, tedious—but straightforward—algebra, gives:

$$\lim_{\omega' \uparrow \omega} \Pi_{\omega'\omega}(\omega'; \omega) - \lim_{\omega' \downarrow \omega} \Pi_{\omega'\omega}(\omega'; \omega) = -\frac{\zeta_1 \zeta_2}{\rho + q + \delta}.$$

Then use the expression for the roots: $\zeta_1 \zeta_2 = -(\rho + q + \delta)/(\sigma^2/2)$. Putting this together proves the result. \square

B.2 Industry Social Planner's Problem

In this section we introduce a dynamic programming problem whose solution gives the equilibrium value for the thresholds $\underline{\omega}, \bar{\omega}$. This problem has the interpretation of a fictitious social planner located in a given industry who maximizes net consumer surplus by deciding how many of the agents currently located in the industry work and how many rest and whether to adjust the number of workers in the industry. The equivalence of the solution to this problem with the equilibrium value of an industry's worker. First, it establishes that our

market decentralization is rich enough to attain an efficient equilibrium, despite the presence of search frictions. Second, it gives an alternative argument to establish the uniqueness of the equilibrium values for the thresholds $\underline{\omega}$ and $\bar{\omega}$. Third, it connects our results with the decision theoretic literature analyzing investment and labor demand model with costly reversibility.

The industry planner maximizes the net surplus from the production of the final good in an industry with current log productivity \tilde{x} and l workers, taking as given aggregate consumption C and aggregate output Y . The choices for this planner are to increase the number of workers located in this industry (hire), paying \bar{v} to the households for each or them, or to decrease the number of workers located in the industry (fire), receiving a payment \underline{v} for each. Increases and decreases are non-negative, and the prices associated with them have the dimension of an asset value, as opposed to a rental. We let $M(\tilde{x}, l)$ be the value function of this planner, hence:

$$M(\tilde{x}, l) = \max_{l_h, l_f} \mathbb{E} \left(\int_0^\infty e^{-(\rho+\delta)t} ((S(\tilde{x}(t), l(t)) + \underline{v}ql(t)) dt - \bar{v}dl_h(t) + \underline{v}dl_f(t)) \middle| \tilde{x}(0) = \tilde{x}, l(0) = l \right)$$

subject to $dl(t) = -ql(t)dt + dl_h(t) - dl_f(t)$ and $d\tilde{x} = \mu_x dt + \sigma_x dz$. (67)

The $l_h(t)$ and $l_f(t)$ are increasing processes describing the cumulative amount of “hiring” and “firing” and hence $dl_h(t)$ and $dl_f(t)$ intuitively have the interpretation of hiring and firing during period t . The term $ql(t)dt$ represent the exogenous quits that happens in a period of length dt . The planner discounts at rate $\rho + \delta$, accounting both for the discount rate of households and for the rate at which her industry disappears.

The function $S(\tilde{x}, l)$ denotes the return function of the industry social planner per unit of time and is given by

$$S(\tilde{x}, l) = \max_{E \in [0, l]} u'(C) \int_0^{Ee^{\tilde{x}}} \left(\frac{Y}{y} \right)^{\frac{1}{\theta}} dy + b_r(l - E) + \delta l \underline{v}.$$

The first term is the consumer’s surplus associated with the particular good, obtained by the output produced by E workers with log productivity \tilde{x} . The second term is value of the workers that the planner chooses to send back to the household, receiving \underline{v} for each. The third term is the value of the “sale” of all the workers if the industry shuts down. Setting $q = \delta = b_r = 0$ our problem is formally equivalent to [Bentolila and Bertola’s \(1990\)](#) model of a firm deciding employment subject to a hiring and firing cost and to [Abel and Eberly’s \(1996\)](#) model of optimal investment subject to costly irreversibility, i.e. a different buying and selling price for capital.

Using the envelope theorem, we find that the marginal value of an additional worker is:

$$S_l(\tilde{x}, l) = \max \left\{ u'(C) \left(\frac{Y(e^{\tilde{x}})^{\theta-1}}{l} \right)^{\frac{1}{\theta}}, b_r \right\} + \delta \underline{v} \quad (68)$$

$$\equiv s \left(\frac{(\theta-1)\tilde{x} + \log Y - \log l}{\theta} + \log u'(C) \right)$$

where the function $s(\cdot)$ is given by $s(\omega) = \max\{e^\omega, b_r\} + \delta \underline{v}$ and is identical to the expression for the per-period value of a worker in our equilibrium, except that $\delta \underline{v}$ is in place of $(q + \delta)\underline{v}$. This is critical to the equivalence between the two problems.

To prove this equivalence, we write the industry social planner's Hamilton-Jacobi-Bellman equation. For each \tilde{x} , there are two thresholds, $\underline{l}(\tilde{x})$ and $\bar{l}(\tilde{x})$ defining the range of inaction. The value function $M(\cdot)$ and thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ solve the Hamilton-Jacobi-Bellman equation if the following two conditions are met:

1. For all \tilde{x} , and $l \in (\underline{l}(\tilde{x}), \bar{l}(\tilde{x}))$ employment decays exponentially with the quits at rate q and hence the value function M solves

$$(\rho + \delta)M(\tilde{x}, l) = S(\tilde{x}, l) - qM_l(\tilde{x}, l) + \mu_x M_{\tilde{x}}(\tilde{x}, l) + \frac{\sigma_x^2}{2} M_{\tilde{x}\tilde{x}}(\tilde{x}, l). \quad (69)$$

2. For all (\tilde{x}, l) outside the interior of the range of inaction,

$$(\rho + \delta)M(\tilde{x}, l) + qlM_l(\tilde{x}, l) - \mu_x M_{\tilde{x}}(\tilde{x}, l) - \frac{\sigma_x^2}{2} M_{\tilde{x}\tilde{x}}(\tilde{x}, l) \leq S(\tilde{x}, l), \quad (70)$$

$$\underline{v} = M_l(\tilde{x}, l) \quad \forall l \geq \bar{l}(\tilde{x}), \text{ and } \bar{v} = M_l(\tilde{x}, l) \quad \forall l \leq \underline{l}(\tilde{x}) \quad (71)$$

Equation (71) is also referred to as *smooth pasting*. Since $M(\tilde{x}, \cdot)$ is linear outside the range of inaction, a twice-continuously differentiable solution implies *super-contact*, or that for all \tilde{x} :

$$0 = M_{ll}(\tilde{x}, \bar{l}(\tilde{x})) = M_{ll}(\tilde{x}, \underline{l}(\tilde{x})). \quad (72)$$

According to Verification Theorem 4.1, Section VIII in **Fleming and Soner (1993)**, a twice-continuously differentiable function $M(\tilde{x}, l)$ satisfying **equations (69), (71), and (72)** solves the industry social planner's problem.

If M is sufficiently smooth, finding the optimal thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ can be stated as a boundary problem in terms of the function $M_l(\tilde{x}, l)$ and its derivatives. To see this,

differentiate both sides of [equation \(69\)](#) with respect to l and replace S_l using [equation \(68\)](#):

$$(\rho + \delta + q)M_l(\tilde{x}, l) = s \left(\frac{(\theta - 1)\tilde{x} + \log Y - \log l}{\theta} + \log u'(C) \right) - qlM_l(\tilde{x}, l) + \mu_x M_{\tilde{x}l}(\tilde{x}, l) + \frac{\sigma_x^2}{2} M_{\tilde{x}\tilde{x}l}(\tilde{x}, l). \quad (73)$$

If the required partial derivatives exist, any solution to the industry social planner's problem must solve [equations \(71\)–\(73\)](#). Moreover, there is a clear relationship between the value function $v(\omega)$ in the decentralized problem and the marginal value of a worker M_l in the industry social planner's problem:

LEMMA 3. Assume that $\theta \neq 1$ and that the functions $M_l(\cdot)$ and $v(\cdot)$ satisfy

$$M_l(\tilde{x}, l) = v(\omega), \text{ where } \omega = \frac{\log Y + (\theta - 1)\tilde{x} - \log l}{\theta} + \log u'(C) \quad (74)$$

and that thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ and the thresholds levels $\{\underline{\omega}, \bar{\omega}\}$ satisfy

$$\log \bar{l}(\tilde{x}) = \log Y + (\theta - 1)\tilde{x} - \theta(\underline{\omega} - \log u'(C)) \quad (75)$$

$$\log \underline{l}(\tilde{x}) = \log Y + (\theta - 1)\tilde{x} - \theta(\bar{\omega} - \log u'(C)). \quad (76)$$

Then, $M_l(\cdot)$ and $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ solve [equations \(71\)–\(73\)](#) for all \tilde{x} and $l \in [\underline{l}(\tilde{x}), \bar{l}(\tilde{x})]$ if and only if $v(\cdot)$ and $\{\underline{\omega}, \bar{\omega}\}$ solve [equations \(12\)](#).

Proof. Differentiate [equation \(74\)](#) with respect to \tilde{x} and l to get

$$M_{l\tilde{x}}(\tilde{x}, l) = v'(\omega) \frac{\theta - 1}{\theta}, \quad M_{l\tilde{x}\tilde{x}}(\tilde{x}, l) = v''(\omega) \left(\frac{\theta - 1}{\theta} \right)^2 \quad \text{and} \quad M_{ll}(\tilde{x}, l) = -v'(\omega) \frac{1}{\theta}.$$

Recall that a solution of [equation \(12\)](#) is equivalent to a solution to [equations \(39\)](#), [\(40\)](#), and $v(\bar{\omega}) = \bar{v}$ and $v(\underline{\omega}) = \underline{v}$. The equivalence between [equation \(12\)](#) and [equations \(71\)–\(73\)](#) is immediate, recalling the definitions of μ and σ . \square

This lemma has important implications. First, it establishes, not surprisingly, that the equilibrium allocation is Pareto Optimal. Second, since the industry social planner's problem is a maximization problem, the solution is easy to characterize. For instance, since the problem is convex, it has at most one solution and hence the equilibrium value of a worker is uniquely defined, for given $u'(C)$ and Y . The fact that v is increasing is then equivalent to the concavity of $S(\tilde{x}, \cdot)$. Finally, notice that [Proposition 1](#) establishes existence and uniqueness

of the solution to [equation \(12\)](#) only under mild conditions on $s(\cdot)$, i.e. that it was weakly increasing and bounded below. [Proposition 1](#) can be used to extend the uniqueness and existence results of the literature of costly irreversible investment to a wider class of production functions. Currently the literature uses that the production function is of the form $x^{a_x} l^{a_l}$ for some constants a_x and a_l , with $0 < a_l < 1$, as in [Abel and Eberly \(1996\)](#). [Proposition 1](#) shows that the only assumption required is that the production function be concave in l , and that the marginal productivity of the factor l can be written as a function of the ratio of the quantity of the input l to (a power of) the productivity shock x .

B.3 Heterogeneous Industries

This section extends the directed search model to include heterogeneity in households' human capital. In equilibrium, industries can be divided into different classes. Industries that attract households with high human capital pay high wages, but the stochastic process for their wages is a scaled version of the one for an industry that attracts households with less human capital. Still, all industries have the same process for the log full employment wage ω (measured in utils) and the same rest and search unemployment rates. This justifies our fixed effect treatment of US industry wage data in [Section 6.1](#).

We prove in this section that in the directed search model with logarithmic utility, the values of the thresholds $\underline{\omega}$ and $\bar{\omega}$ are the same across industries, although the level of consumption, and hence the wage in units of goods, is different. We omit a proof of a similar result in the random search model, under the assumption that workers with a particular human capital level contact other workers with the same human capital level at rate α , at which point they may join the workers' industry.

We turn now to a description of the directed search model. Households are indexed by one of K human capital types, denoted by h_k satisfying $0 < h_1$ and $h_k < h_{k+1}$, for $k = 1, 2, \dots, K-1$, with $h_K = 1$. For notational convenience, let $h_0 = 0$ and $\Delta h_k \equiv h_k - h_{k-1}$. Let H_k denote the cumulative distribution of households' human capital types, so that there are H_k households with human capital $h_j \leq h_k$, and there are $\Delta H_k \equiv H_k - H_{k-1}$ household with human capital type h_k for $k = 1, \dots, K$.

Recall that industries are indexed by j which belong to $[0, 1]$. The meaning of type h_k human capital is that such household can work in any industry labeled $j \in (0, h_k]$. Assume

$$\frac{\Delta H_{k+1}}{\Delta h_{k+1}} < \frac{\Delta H_k}{\Delta h_k}, \quad (77)$$

for $k = 1, \dots, K-1$. We then look for an equilibrium where type h_k households work in industries of type $j \in (h_{k-1}, h_k]$. In this equilibrium, we talk of both household and industries

of type k . For workers to sort themselves across industries in this way, it must be the case that wages are increasing in industry type, and [equation \(77\)](#) insures that labor supply is in fact decreasing in industry type.

Let L_k denote the fraction of members of type k households who are located in type k industries and $L_{0,k}$ denote the fraction located in newly created industries within the k class. Thus $L_k(\Delta H_k/\Delta h_k)$ is the number of household members per type k industry, either working or in rest unemployment.

Households with different human capital have different consumption, and hence different marginal utility. Letting C_k be the consumption per household for those with human capital k , we have that the log full-employment wage for household of type k follows:

$$\omega_k(t) \equiv \frac{\log Y + (\theta - 1) \log x(t) - \log l(t)}{\theta} + \log u'(C_k) \quad (78)$$

where Y is aggregate output, $x(t)$ is industry productivity, and $l(t)$ is the number of workers in the industry. We characterize an equilibrium where the process for ω_k is identical for all k .

PROPOSITION 7. Assume log utility, $u(C) \equiv \log C$, and that [equation \(77\)](#) holds. Let $(L^*, \underline{\omega}^*, \bar{\omega}^*)$ be the equilibrium values for the model without heterogeneity. Then there is an equilibrium of the model with heterogeneity with $(L_k, \bar{\omega}_k, \underline{\omega}_k) = (L^*, \bar{\omega}^*, \underline{\omega}^*)$ for all k and

$$\frac{C_k}{C_{k'}} = \left(\frac{\Delta h_k \Delta H_{k'}}{\Delta H_k \Delta h_{k'}} \right)^{1/\theta}.$$

Proof. For the processes $\{\omega_k(t)\}$ to be identical across industries, the difference in the log of the marginal utilities must be compensated by a difference in the level of the employment per industry, so that any two industries in classes k and k' created at the same time and with the same history of shocks have employment l_k and $l_{k'}$ satisfying

$$\log l_k(t) - \log l_{k'}(t) = \theta(\log u'(C_k) - \log u'(C_{k'})).$$

Aggregating across shocks and using the logarithmic utility assumption and the conjecture about the nature of equilibrium, the number of workers located in type k industries is

$$\frac{L^* \Delta H_k C_k^\theta}{\Delta h_k} \equiv \beta \quad (79)$$

for all $k = 1, \dots, K$ and some constant β .

The distribution f evaluated at the upper bound still satisfies

$$\frac{\sigma^2}{2} f'(\bar{\omega}) - \left(\mu + \frac{\theta\sigma^2}{2} \right) f(\bar{\omega}) = \delta \frac{L_0^*}{L^*}, \quad (80)$$

where L_0^* is the fraction of workers in a new industry, independent of k in the proposed equilibrium. The requirement that the log full-employment wages is $\bar{\omega}$ in new industries implies

$$\frac{L_0^* \Delta H_k C_k^\theta}{\Delta h_k} = Y x_0^{\theta-1} e^{-\theta\bar{\omega}}. \quad (81)$$

From [equation \(79\)](#), the left hand side is $\beta L_0^*/L^*$. Eliminate L_0^*/L^* using [equation \(80\)](#) to get

$$\beta = \phi_1 Y, \text{ where } \phi_1 \equiv \frac{\delta x_0^{\theta-1} e^{-\theta\bar{\omega}}}{\frac{\sigma^2}{2} f'(\bar{\omega}) - \left(\mu + \frac{\theta\sigma^2}{2} \right) f(\bar{\omega})}. \quad (82)$$

In each industry class k we can solve for the productivity consistent with (l, ω, Y, C_k) as:

$$x = \xi(l, \omega, Y, C_k) \equiv \left(\frac{l e^{\theta\omega} C_k^\theta}{Y} \right)^{\frac{1}{\theta-1}}. \quad (83)$$

Then using the production function, output in a industry in such an industry class, with l workers and log full-employment wage ω , is

$$Q(l, \xi(l, \omega, Y, C_k)) = Y^{\frac{-1}{\theta-1}} (e^\omega l C_k)^\frac{\theta}{\theta-1} \min\{1, e^\omega/b_r\}^\theta. \quad (84)$$

Using this notation, we can write the analog of [equation \(48\)](#) as

$$\begin{aligned} Y &= \left(\sum_{k=1}^K \int_{h_{k-1}}^{h_k} Q(l(j, t), \xi(l(j, t), \omega(j, t), Y, C_k))^\frac{\theta-1}{\theta} dj \right)^\frac{\theta}{\theta-1} \\ &= \left(\sum_{k=1}^K \int_{h_{k-1}}^{h_k} Q \left(\frac{L^* \Delta H_k}{\Delta h_k}, \xi \left(\frac{L^* \Delta H_k}{\Delta h_k}, \omega(j, t), Y, C_k \right) \right)^\frac{\theta-1}{\theta} \frac{l(j, t)}{\frac{L^* \Delta H_k}{\Delta h_k}} dj \right)^\frac{\theta}{\theta-1}. \end{aligned}$$

The second equation follows because $Q(\cdot, \xi(\cdot, \omega, Y, C_k))^\frac{\theta-1}{\theta}$ is linear in l ([equation 84](#)). To solve this, we change the variable of integration from the name of the industry j to its log full-employment wage ω and number of workers l . Let $\tilde{f}(\omega, l)$ be the density of the joint invariant distribution of workers in industries (ω, l) , as discussed in [Appendix A.4](#). Notice

that under our hypothesis this distribution is the same for all k . Then

$$Y = \left(\sum_{k=1}^K \Delta h_k \int_{\underline{\omega}}^{\bar{\omega}} \int_0^{\infty} Q \left(\frac{L^* \Delta H_k}{\Delta h_k}, \xi \left(\frac{L^* \Delta H_k}{\Delta h_k}, \omega, Y, C_k \right) \right)^{\frac{\theta-1}{\theta}} \frac{l}{\frac{L^* \Delta H_k}{\Delta h_k}} \tilde{f}(\omega, l) dl d\omega \right)^{\frac{\theta}{\theta-1}}.$$

Since $f(\omega) = \int_0^{\infty} \frac{l \Delta h_k}{L^* \Delta H_k} \tilde{f}(\omega, l) dl$, we can solve the inner integral to obtain

$$Y = \left(\sum_{k=1}^K \Delta h_k \int_{\underline{\omega}}^{\bar{\omega}} Q \left(\frac{L^* \Delta H_k}{\Delta h_k}, \xi \left(\frac{L^* \Delta H_k}{\Delta h_k}, \omega, Y, C_k \right) \right)^{\frac{\theta-1}{\theta}} f(\omega) d\omega \right)^{\frac{\theta}{\theta-1}},$$

without characterizing the joint density \tilde{f} . Using [equation \(84\)](#) and simplifying,

$$Y = L^* \left(\sum_{k=1}^K \Delta H_k C_k \right) \left(\int_{\underline{\omega}}^{\bar{\omega}} e^{\omega} \min\{1, e^{\omega}/b_r\}^{\theta-1} f(\omega) d\omega \right). \quad (85)$$

Since total output in the economy is consumed by the households,

$$Y = \sum_{k=1}^K \Delta H_k C_k. \quad (86)$$

Then [equation \(85\)](#) implies

$$L^* = \left(\int_{\underline{\omega}}^{\bar{\omega}} e^{\omega} \min\{1, e^{\omega}/b_r\}^{\theta-1} f(\omega) d\omega \right)^{-1}. \quad (87)$$

This defines L^* . Next, substitute for C_k in [equation \(86\)](#) using [equation \(79\)](#):

$$Y^{\theta} = \frac{\beta}{L^*} \left(\sum_{k=1}^K \Delta H_k^{\frac{\theta-1}{\theta}} \Delta h_k^{\frac{1}{\theta}} \right)^{\theta}. \quad (88)$$

Eliminate β using [equation \(82\)](#) to get an expression for total output.

$$Y = \left(\frac{\phi_1}{L^*} \right)^{\frac{1}{\theta-1}} \left(\sum_{k=1}^K \Delta H_k^{\frac{\theta-1}{\theta}} \Delta h_k^{\frac{1}{\theta}} \right)^{\frac{\theta}{\theta-1}}. \quad (89)$$

This defines Y . Finally, one can go back to [equation \(82\)](#) to determine β and then to [equation \(79\)](#) to pin down C_k , closing the model. Note that assumption [\(77\)](#) implies consumption is increasing in k .

To prove that a type k household prefer to work on industry k to other industries $j = 1, \dots, k - 1$, we show that wages are increasing in k . This follows because, with logarithmic utility, the actual wage is the product of ω , whose distribution is independent of k , and consumption. \square